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*The Ten Nodes of the Rational Sextic and of the Cayley Symmetroid.**

By Arthur B. Coble.†

Introduction.

The general rational plane sextic with ten nodes occupies a unique position among all rational plane curves in that it is the rational curve of lowest order which can not be transformed by ternary Cremona transformation into a straight line, that is to say its order can not be reduced by such transformation. It may, however, be transformed into other rational sextics, and this can be accomplished by Cremona transformations of infinitely many distinct types. One of the principal results of this paper is that the sextic and all of its sextic transforms are comprised under precisely $2^{13}.31.51$ projectively distinct types

The intimate relation between the ten nodes of a rational plane sextic and the ten nodes of that quartic surface known as the Cayley symmetroid has been pointed out by J. R. Conner. ‡ It is not surprising therefore to find that a similar fact is true of the symmetroid under regular Cremona transformation in space.

The methods of investigation here employed have been set forth in an earlier series of papers by the writer. § Some of the points of view may be recapitulated briefly as follows. We shall be interested in a Cremona transformation C only in so far as it disturbs projective relations so that for our purposes $C \equiv \pi C \pi'$ where π, π' are arbitrary projectivities. If C has the singular points, or F-points, $p_1, \ldots, p_\rho$, and C^{-1} the F-points $q_1, \ldots, q_\rho$ then C transforms curves of order x_0 and multiplicities x_i at p_i into curves of order x'_0 and multiplicities x'_j at q_j $(i, j=1, \ldots, \rho)$ where x' is determined in terms of x by the linear transformation, $L(C)$,

$$(1) \qquad L(C): x'_0 = mx_0 - \Sigma_{i=1}^{i=\rho} r_i x_i, \quad x'_j = s_j x_0 - \Sigma_{i=1}^{i=\rho} \alpha_{ij} x_i.$$

* Read by title at the meeting of the Chicago Section of the American Mathematical Society, April, 1919.

† This investigation has been carried on under the auspices of the Carnegie Institution of Washington, D. C.

‡ "The Rational Sextic Curve and the Cayley Symmetroid," this Journal, Vol. XXXVII (1915), p. 29.

§ "Point Sets and Cremona Groups," Part II, *Trans. Amer. Math. Soc.*, Vol. XVII (1916), p. 345; referred to hereafter as P. S. II.

In (1) the coefficients are m, the order of C; r_i, the order of F-point p_i; s_j, the order of the F-point q_j; and α_{ij}, the number of times the fundamental curve, or F-curve, of p_i passes through q_j.

The product CC' of two Cremona transformations can be unique only when the position of the F-points p'_i of C' with respect to the F-points q_j of C^{-1} is definitely specified. In order to limit the possibilities which arise in this connection we require that the points p_i shall be in a set of n points P^2_n and the points q_j in a set of n points Q^2_n such that the further pairs $p_{\rho+1}$, $q_{\rho+1}$, ..., p_n, q_n are pairs of ordinary corresponding points of C. This amplifies the linear transformation $L(C)$ by the equations

$$(2) \qquad x'_l = -(-1)x_l \quad (l=\rho+1, \ldots, n),$$

and the two sets P^2_n, Q^2_n are called *congruent under* C. In forming the product CC' we require that the points of P'^2_n shall coincide *in some order* with the points of Q^2_n. This possibility of reordering the points of a set—a non-projective operation for $n>4$—is accounted for by adjoining to the linear transformations $L(C)$ those additional ones constituting a $g_{n!}$ which permute the variables $x_1, \ldots, x_n$. Thus the operations involved in passing from a set P^2_n to all sets Q^2_n congruent in some order to P^2_n—operations which constitute a group $G_{n,2}$—are reflected by simple isomorphism in the transformations $L(C)$ of the group $g_{n,2}$ generated by $g_{n!}$ and the transformation $L(C)$ determined by a single quadratic transformation C, since the general Cremona transformation is a product of properly ordered quadratic transformations. Obviously when a set P^2_n is in question this general transformation is restricted to have $\rho \overline{<} n$ F-points.*

We are concerned here with the set P^2_{10} of the nodes of a rational plane sextic and can state at once the theorem

(3) *A sextic S with nodes P^2_{10} can be transformed into a sextic $\overline{S}$ with nodes Q^2_{10} by ternary Cremona transformation if and only if the sets P^2_{10} and Q^2_{10} are congruent.*

For if S is transformed by C into $\overline{S}$ the ρ F-points of C must be all within P^2_{10} else the order of the transform is greater than 6. Hence $\rho \overline{<} 10$. If $\rho<10$ the nodes of S in P^2_{10} which are ordinary points of C pass into nodes of $\overline{S}$ in the congruent set Q^2_{10}.

The arithmetic group $g_{n,2}$ simply isomorphic with $G_{n,2}$ has integer coefficients. We shall prove in § 1 that there is only a finite number of projectively distinct sets Q^2_{10} congruent to the set P^2_{10} *when P^2_{10} is the set of nodes of S*, and that, for all the operations of $G_{10,2}$ whose isomorphic elements in

* These remarks are amplified in P. S. II, § 1.

$g_{10,2}$ have coefficients congruent modulo 2 to those of the identity, the set Q^2_{10} is projective to P^2_{10} and therefore may be made to coincide with P^2_{10} by a subsequent projectivity. These elements form an invariant subgroup $\bar{g}_{10,2}$ of $g_{10,2}$ whose factor group $g^{(2)}_{10,2}$* is finite and of order $10!\,2^{13}.31.51$.

An important problem is now apparent. Since $g_{10,2}$ is infinite and discontinuous (P. S. II, § 4 (18)) and $\bar{g}_{10,2}$ is of finite index under $g_{10,2}$ there follows that an infinite discontinuous Cremona group $\bar{G}_{10,2}$ exists which transforms the sextic S into itself. $\bar{G}_{10,2}$ also will contain an invariant subgroup $\bar{\bar{G}}_{10,2}$ which consists of those elements of $\bar{G}_{10,2}$ for which every point of S is fixed. It may be and probably is true that $\bar{\bar{G}}_{10,2}$ is merely the identical transformation, but in any case the factor group of $\bar{\bar{G}}_{10,2}$ under $\bar{G}_{10,2}$ will be represented by a discontinuous group of elements of the form

$$t' = \frac{at+b}{ct+d},$$

where t is the parameter on the rational curve S. From certain geometrical considerations it seems reasonable to think that this discontinuous group is of genus 4, and that the ten nodes of S can be expressed by means of Riemannian modular functions of genus 4.

The ten nodes P^3_{10} of the Cayley symmetroid Σ, discussed in Part II, behave under *regular*† Cremona transformations in space much like the ten nodes of S under ternary transformation. One novelty introduced in § 4 is the *dilation* of the regular group in a space S_k into a subgroup of the regular group in a higher space S_{k+l}.

PART I.

The Ten Nodes P^2_{10} of the Sextic S.

§ 1. *The Equivalence of the f-curves of P^2_{10} under $\bar{G}_{10,2}$.*

The first theorem which we shall use is

(4) *The group $\bar{G}_{10,2}$ which leaves the sextic S unaltered is generated by the involutions conjugate under $G_{10,2}$ to the Bertini involution.*

We recall that the Bertini involution is defined as follows. Given eight points $p_1, \ldots, p_8$ in the plane, the ∞^3 sextics with nodes at these points have the property that the ∞^2 sextics of the system on a point x pass also through another point y, the copoint of x in the involution B. Obviously every sextic

* The factor groups $g^{(2)}_{n,k}$ for the group $g_{n,k}$ have been identified with known groups in the author's paper entitled "Theta Modular Groups Determined by Point Sets," this Journal, Vol. XL (1918), p. 317; cited hereafter as T. M. Groups. This paper emphasizes the geometric possibilities of the particular cases $g_{2p+2,p}$. It is of interest to find that other cases also have geometric applications.

† Cf. P. S. II, § 4, or § 4 of this paper for the definition.

of the system is a fixed curve, and every additional node of such a sextic is a fixed point of the involution whence it leaves the sextic S with nodes at $p_1, \ldots, p_8, p_9, p_{10}$ unaltered. By permutation of the points of P_{10}^2 all the $\binom{10}{8}$ Bertini involutions attached to the set P_{10}^2 are obtained. Moreover, if C is any Cremona transformation with F-points at P_{10}^2, then CBC^{-1} also leaves S unaltered. For C transforms S into a sextic S' with nodes at Q_{10}^2, B leaves S' unaltered, and C^{-1} transforms S' back into S. Hence the conjugate set of involutions described in (4) all belong to $\overline{G}_{10,2}$. The proof that they generate $\overline{G}_{10,2}$ will appear later. Meanwhile two objects conjugate under $\overline{G}_{10,2}$ will be called equivalent, and this relation of equivalence will be denoted by the symbol $\equiv$.

The f-curves of the set P_{10}^2 are the transforms by Cremona transformation of the sets of directions about the points. Instead of the general Cremona transformation we may make repeated use of the quadratic transformation $A_{j_1 j_2 j_3}$ with F-points at $p_{j_1}, p_{j_2}, p_{j_3}$. Beginning then with the set of directions about p_1, it becomes under the $g_{n!}$ of permutations of the points, a set of directions about any one of the ten points. Applying A_{123} to the set of directions at p_1 it becomes the line on $q_2 q_3$, and under $g_{n!}$ this becomes any line $q_i q_j$. Applying A_{123} to the line $p_4 p_5$ it becomes a conic on $q_1 q_2 q_3 q_4 q_5$. Proceeding in this way the totality of f-curves of the set P_{10}^2 is obtained. We shall denote by its *signature*, $f_r(j_1^{k_1}, j_2^{k_2}, \ldots, j_{10}^{k_{10}})$, an f-curve of order r with multiple points of orders $k_1, \ldots, k_{10}$ at the points $p_1, \ldots, p_{10}$, respectively. A systematic derivation of the types of f-curves is carried out in the following table (5):

(5)

f-curve	operated upon by	becomes	which is	
$f_0(1)$	A_{123}	$f_1(23)$*		
	A_{234}	$f_0(1)$		
$f_1(23)$	A_{123}	$f_0(1)$		
	A_{124}	$f_1(23)$		
	A_{145}	$f_2(12345)$*		
$f_2(12345)$	A_{345}	$f_1(12)$		
	A_{456}	$f_2(12345)$		
	A_{567}	$f_3(12345^2 67)$*		
	A_{678}	$f_4(123456^2 7^2 8^2)$	$\equiv f_2(12345)$	(1^0)
$f_3(12345^2 67)$	A_{567}	$f_2(12345)$		
	A_{467}	$f_3(12345^2 67)$		
	A_{578}	$f_3(12345^2 67)$		
	A_{678}	$f_4(12345^2 6^2 7^2 8)$	$\equiv f_2(12348)$	(1^0)
	A_{589}	$f_4(12345^3 6789)$*		
	A_{789}	$f_5(12345^2 67^3 8^2 9^2)$	$\equiv f_3(12345^2 67)$	(2^0)
	A_{8910}	$f_6(12345^2 678^3 9^3 10^3)$	$\equiv f_4(1234678^3 910)$	(3^0)
$f_4(12345^3 6789)$	A_{567}	$f_3(12345^2 89)$		
	A_{678}	$f_5(12345^3 6^2 7^2 8^2 9)$	$\equiv f_3(123456^2 9)$	(2^0)
	A_{5610}	$f_4(12345^3 6789)$		
	A_{6710}	$f_6(12345^3 6^3 7^3 8910^2)$	$\equiv f_4(12345^3 6789)$	(3^0)

New types of f-curves as they are obtained are starred, and these new types are in turn subjected to transformation. However, as the process goes on, the new types obtained are equivalent under $\bar{G}_{10,2}$ to earlier types and these need not be transformed afresh.

In order to prove the equivalences (1^0), (2^0), (3^0) listed in the table (5), and at the same time to verify that the two further equivalences

$$(4^0) \qquad f_3(i_1i_2i_3i_4i_5i_6j^2) \equiv f_3(i_1i_2i_3i_4i_5i_6k^2),$$

$$(5^0) \qquad f_4(i_1i_2i_3i_4i_5i_6i_7jk^3) \equiv f_4(i_1i_2i_3i_4i_5i_6i_7j^3k),$$

are valid we begin with the equivalence,

$$(6) \qquad f_0(i) \equiv f_6(i^3j_1^2j_2^2j_3^2j_4^2j_5^2j_6^2j_7^2)$$

which is derived at once from a Bertini involution. If the two members of this equivalence be transformed by C then the two transforms are themselves equivalent under the transform of the Bertini involution by C, whence according to (4) they are equivalent under $\bar{G}_{10,2}$. Transforming (6) by $A_{ij_1j_2}$, $A_{ij_3j_4}$, $A_{ij_5j_6}$ and $A_{ij_7j_8}$ successively we get

$$(7) \qquad f_1(j_1j_2) \equiv f_5(i^2j_1j_2j_3^2j_4^2j_5^2j_6^2j_7^2),$$

$$(1^0) \qquad f_2(ij_1j_2j_3j_4) \equiv f_4(ij_1j_2j_3j_4j_5^2j_6^2j_7^2),$$

$$(4^0) \qquad f_3(i^2j_1j_2j_3j_4j_5j_6) \equiv f_3(j_1j_2j_3j_4j_5j_6j_7^2),$$

$$(5^0) \qquad f_4(i^3j_1j_2j_3j_4j_5j_6j_7j_8) \equiv f_4(ij_1j_2j_3j_4j_5j_6j_7^3j_8).$$

If now we transform (4^0) by $A_{j_1j_7j_8}$ and $A_{j_7j_8j_9}$ we get

$$(2^0) \qquad f_5(i^2j_1^3j_2j_3j_4j_5j_6j_7^2j_8^2) \equiv f_2(j_1j_2j_3j_4j_5j_6j_7^2),$$

$$(3^0) \qquad f_6(i^2j_1j_2j_3j_4j_5j_6j_7^3j_8^3j_9^3) \equiv f_4(j_1j_2j_3j_4j_5j_6j_7^3j_8j_9),$$

whence all the equivalences used in limiting the table (5) have been established.

A glance at the list of equivalences established shows that the signatures of equivalent f-curves are congruent modulo 2, and further that no two of the non-equivalent f-curves in the first column of table (5) have signatures which are congruent modulo 2. This is to be expected since the signatures, $f_{r_i}(p_1^{a_{i1}}, p_2^{a_{i2}}, \ldots, p_{10}^{a_{i10}})$, of the f-curves of P_{10}^2 arise from the columns other than the first of the matrices of the linear transformations L of (1) and (2), and the transformation L which correspond to the Bertini involutions (and therefore also to the conjugates of the Bertini involutions) are congruent to the identity modulo 2. Thus we have proved that

(8) *Under the group generated by the conjugate set of involutions which contains a Bertini involution, the infinite number of f-curves of P_{10}^2 divide into $527 = 2^{p-1}(2^p+1)-1\,(p=5)$ sets such that the infinite number in*

any one set are equivalent and that the f-curves from different sets are not equivalent. Equivalent f-curves have signatures congruent modulo 2. *As types of these sets we may take the* $\binom{10}{1}$ *of form* $f_0(i)$, *the* $\binom{10}{2}$ *of form* $f_1(i_1 i_2)$, *the* $\binom{10}{5}$ *of form* $f_2(i_1 i_2 i_3 i_4 i_5)$, *the* $\binom{10}{6}$ *of form* $f_3(i_1^2 i_2 i_3 i_4 i_5 i_6 i_7)$, *and the* $\binom{10}{9}$ *of form* $f_4(i_1^3 i_2 i_3 i_4 i_5 i_6 i_7 i_8 i_9)$.

Since all f-curves with signatures congruent modulo 2 are equivalent under the conjugates of a Bertini involution, there follows that the subgroup $g(2)$ of $g_{10,2}$ which is congruent modulo 2 to the identity is generated by these conjugates. Now the index of $g(2)$ under $g_{10,2}$ is the order of the finite group of permutations, $g_{10,2}^{(2)}$, of the above 527 sets. The order of this group has been determined in "T. M. Groups." In fact the signatures of the f-curves reduce modulo 2 to the coefficients of the forms b_1, c_2 of the table there given (p. 323 for $\nu=\kappa=2$). They are permuted like the even characteristics of the theta functions for $p=5$ under the group (p. 337 *loc. cit.*) of order

$$\mu=2^{21}(2^5-1)(2^8-1)(2^6-1)(2^4-1)(2^2-1),$$

which leaves one even theta characteristic unaltered. Now $g(2)$ is simply isomorphic either with $\overline{G}_{10,2}$ or with a subgroup of it. In the first case μ divided by $10!$ (to account for the mere ordering of the set P_{10}^2) will be the number of sextics projectively distinct from S and including S itself. In the second case this number will be a smaller factor of $\mu/10!$ Now assuming that μ is the proper index of $\overline{G}_{10,2}$ under $G_{10,2}$ then the index μ' of $\overline{G}_{9,2}$ under $G_{9,2}$ (where these new groups are defined precisely as the groups $\overline{G}_{10,2}$ and $G_{10,2}$ except that all Cremona transformations employed are to have p_{10} as an ordinary point*) is $\mu/527$, since p_{10} or $f_0(10)$ is to be unaltered. Then the number of projectively distinct sets P_9^2 which can be obtained by Cremona transformation from the nine nodes of a sextic is $\mu/527$ divided by $9!$ But according to P. S. II (47) $\mu/9!527=2^8.960$ is precisely this number of sets P_9^2. Hence μ is the index of $\overline{G}_{10,2}$ under $G_{10,2}$ and $\overline{G}_{10,2}$ is generated by the conjugates of the Bertini involution. We have thus completed the proof of (4) and have also proved that

(9) *A rational plane sextic with ten nodes can be transformed by Cremona transformation into precisely* $2^{13}.31.51$ *projectively distinct sextics. Under such transformation these projectively distinct types (with*

* It is proved in P. S. II, § 6, that the generators of $\overline{G}_{9,2}$ are conjugates of Bertini involutions whence all the Cremona transformations with F-points at P^2_9 for which P^2_9 is congruent to itself will leave unaltered the 10-th node of a sextic with nodes at P^2_9.

ordered nodes) are permuted according to the finite group of odd and even theta characteristics for $p=5$, which leaves an even characteristic unaltered. The infinite discontinuous group $\overline{G}_{10,2}$ of Cremona transformations which leaves S unaltered is simply isomorphic with the subgroup $\overline{g}_{10,2}$ of $g_{10,2}$, which is congruent to the identity modulo 2.

§ 2. *The Discriminant Conditions for P_{10}^2.*

In P. S. II, § 8, the set P_7^2 was discussed in connection with the general plane quartic and the sixty-three factors of the discriminant of this quartic arose from the conditions that two points of P_7^2 should coincide, that three should be on a line, and that six should be on a conic. In all these cases an f-curve passes through one more point of the set than is true in general. The conditions might be indicated thus:

$$f_0(i_1i_2)=0 \quad f_1(i_1i_2i_3)=0, \quad \text{and} \quad f_2(i_1i_2i_3i_4i_5i_6)=0.$$

Similarly for the set P_6^2 (P. S. III (1917), § 1), the same conditions give rise to the thirty-six factors of the discriminant of the cubic surface which is mapped from the plane by cubic curves on P_6^2. We shall therefore continue to refer to such conditions as *discriminant conditions for the set,* even though for sets beyond P_8^2 the word discriminant does not have its usual meaning.

For a general set P_{10}^2 the number of these discriminant conditions is infinite, but they all arise from any one—say $f_0(12)=0$—by Cremona transformation. On the other hand when P_{10}^2 is the special set of ten nodes of a sextic S and therefore subject to three conditions, the existence of one discriminant condition—a fourth condition on P_{10}^2—taken together with the three conditions already implied by the existence of S entails the existence of infinitely many discriminant conditions. For example, reverting to the table (5) of § 1, let us begin with the condition $f_0(1,9)=0$ which indicates the existence of a tacnode due to the coincidence in some direction of the nodes p_1, p_9. Transforming this by A_{123} we get the condition $f_1(239)=0$ which expresses that the nodes p_2, p_3, p_9 are on a line. Transforming this by A_{145} we get the condition $f_2(123459)=0$, and this, transformed by A_{678} gives rise to $f_4(123456^27^28^29)=0$. But according to the equivalence (1°) there is a transformation of $\overline{G}_{10,2}$ which leaves the nodes of S unaltered and transforms $f_4(123456^27^28^2)$ into $f_2(12345)$. Therefore if $f_4(123456^27^28^29)=0$ then also $f_2(123459)=0$. Proceeding thus we find that the equivalences of f-curves under $\overline{G}_{10,2}$ imply the identity of corresponding discriminant conditions and we can prove at once by the foregoing methods the theorem:

(10) *The number of discriminant conditions—infinite for the general point set P_{10}^2—is finite for the P_{10}^2 of nodes of S, a set subject to three conditions and containing nine absolute constants. Any two discriminant conditions whose signatures are congruent modulo 2 impose the same* FOURTH *condition on the ten nodes. The $\binom{10}{2}$ conditions of type $f_0(i_1 i_2)=0$, the $\binom{10}{3}$ of type $f_1(i_1 i_2 i_3)=0$, the $\binom{10}{6}$ of type $f_2(i_1 i_2 i_3 i_4 i_5 i_6)=0$, the $\binom{10}{7}$ of type $f_3(i_1^2 i_2 i_3 i_4 i_5 i_6 i_7 i_8)=0$, and the $\binom{10}{10}$ of type $f_4(i_1^3 i_2 i_3 i_4 i_5 i_6 i_7 i_8 i_9 i_{10})=0$, $496=2^{p-1}(2^p-1)$ $(p=5)$ in all, exhaust the number of independent discriminant conditions. The members of this finite set of conditions are permuted under Cremona transformation like the odd theta characteristics under the group of* § 1 (9).

In fact these conditions correspond to the forms b_2, c_8 of the table cited above from T. M. Groups.

From any equivalence there will follow a theorem concerning a special sextic S. Thus from (4°) and (5°) of § 1 we have

(11) *If there exists a cubic curve on seven nodes of S with a double point at one of the three remaining nodes (one condition on S) then there will exist a cubic curve on the same seven nodes and with a double point at* ANY *one of the three remaining nodes.*

(12) *If there exists a quartic curve with triple point at one node of S and on the other nodes, then there will exist a quartic with a triple point at any one node and on the other nodes.*

Part of the content of theorem (10) has been stated by Miss Hilda Hudson,* and her method (Section 4, *loc. cit.*) of proving the equivalence of discriminant conditions is interesting. Unfortunately much of this paper is colored by the false assumption that the rational sextic with which she begins, and which has six nodes on a conic is a general rational sextic with nine absolute constants. Miss Hudson uses a space sextic of genus 4—the complete intersection of a quadric and a cubic surface—and assigns to it four actual nodes by making the cubic touch the quadric at four points, and projects it from an arbitrary point of space. Now if λ, μ are the binary parameters of the generators on the quadric, the sextic of genus 4 has the equation $(a\lambda)^3(b\mu)^3=0$ with fifteen constants. Of these six can be removed by projectivities on λ, μ whence the curve has nine absolute constants. These are in

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* "The Cremona Transformations of a Certain Plane Sextic," *Proceedings of the London Mathematical Society*, Ser. 2, Vol. XV (1916–17), p. 385.

fact its Riemannian moduli since the curve is normal. The four node requirement reduces the number of constants to 5, and projection from an arbitrary point introduces three more, so that the resulting rational sextic has but eight absolute constants and is subject to the further condition that six nodes are on a conic—a well-known condition on the nodes of any projection of the general space sextic of genus 4. The general rational plane sextic should be obtained as the projection of a general rational space sextic, and the latter sextic does not lie on a quadric.

In the same volume of the *Proceedings* Mr. J. Hodgkinson* shows that there can be *at most* thirty rational sextics with nine properly assigned nodes. As a matter of fact this number is exactly twelve.

In view of these misconceptions it may be worth while to develop in some detail the conditions on the nodes of a rational sextic.† Let then $p_1, \ldots, p_8$ be eight general points of the plane with eight absolute constants. They are the base points of a pencil of cubics $C_\lambda = \lambda_1 C_1 + \lambda_2 C_2$ which meet again in a 9-th point P. This is of course a general pencil of cubics, and all of its members are nondegenerate and all are elliptic except for the twelve nodal cubics of the pencil with nodes at $D_1, \ldots, D_{12}$. The net of sextics, $\mu_1 C_1^2 + \mu_2 C_1 C_2 + \mu_3 C_2^2$, has nodes at $p_1, \ldots, p_8$ and is merely the aggregate of pairs of the pencil C_λ. Other sextics with these nodes exist. Such for example is the degenerate sextic $f_1(12) \cdot f_5(123^2 \ldots 8^2)$ whose factors are known to exist and to be unique. Moreover, this sextic is not found in the above net since it is not a pair of cubics of the pencil C_λ. Let then Σ be any sextic, not included in the net, which has double points at $p_1, \ldots, p_8$. The web of sextics

$$\mu_1 C_1^2 + \mu_2 C_1 C_2 + \mu_3 C_2^2 + \mu_4 \Sigma \tag{13}$$

contains *all* sextics with nodes at $p_1, \ldots, p_8$. For if another sextic Σ' not contained in the system (13) should exist, the system of ∞^4 sextics obtained by adjoining Σ' would cut the line $f_1(12)$ in ∞^2 variable pairs and a pencil would have the fixed factor $f_1(12)$ and the variable factor $f_5(123^2 \ldots 8^2)$ contrary to the fact that this quintic is unique.

All the sextics of the web (13) on a point x pass through a second point y, and x, y are partners in the Bertini involution B.‡ In fact, if C_1 is the cubic

* "The Nodal Points of a Plane Sextic," *loc. cit.*, p. 343.

† Cf. E. C. Valentiner, *Tidsskrift for Math.*, Ser. 4, Vol. V (1881), p. 88, and G. Halphen, *M. S. F. Bull.*, Vol. X (1882), p. 162.

‡ Cf. V. Snyder, "The Involutorial Birational Transformation of the Plane of Order Seventeen," this Journal, Vol. XXXIII (1911), p. 327.

of the pencil C_λ on x, then C_1^2 and C_1C_2 are two independent sextics on x; let Σ be a third. These sextics all meet at the intersections of C_1 and Σ. Let the elliptic parameters on C_1 of $p_1, \ldots, p_8$ be $u_1, \ldots, u_8$ (with $u+u'+u''\equiv 0$ as the linear condition) and let u_x, u_y be those of x, y. Then

$$2(u_1+\ldots+u_8)+u_x+u_y\equiv 0.$$

Hence x, y are on a line with the point $u=2(u_1+\ldots+u_8)$ and this is the tangential point of the four points $u=-(u_1+\ldots+u_8)+\frac{\omega}{2}$. If $\frac{\omega}{2}$ is the zero half-period, this is the 9-th base point P; if $\frac{\omega}{2}$ is a proper half-period we we may call the points the *three half-period points on* C_1. Hence a construction for B is as follows: At P, a base point of the pencil C_λ, draw a tangent to the cubic C_λ to meet the cubic C_λ at P_λ, and from P_λ project the cubic into itself to obtain the pairs x, y of B. One easily verifies that the locus of P_λ is a rational quartic on $p_1, \ldots, p_8$ with triple point at P whose tangents are those of cubics with flexes at P. The construction for y is indeterminate only when x is at p_1, or p_2, or $\ldots$, or p_8. The sextic S_{p_1} with triple point at p_1 and nodes at $p_2, \ldots, p_8$ exists and is unique (as is proved at once by reducing its order by a quadratic transformation), and, if x is at any point of S_{p_1}, y is at p_1. Hence B has eight six-fold F-points p_i with corresponding f-curve S_{p_i} and is of order 17. Evidently every sextic (13) and every cubic C_λ is a fixed curve of B.

We are interested primarily in the fixed points of B. These occur at the point P and at the three half-period points on C_λ. The latter run over a locus N which has triple points at p_i with the same tangents as S_{p_i} since these three directions at p_i are self corresponding. Also N is of order 9 since a cubic C_λ meets it in three points outside the eight points p_i. The fixed point P and the fixed point p_9—a general point on N—are of different kinds. P is a fixed point with fixed directions, *i. e.*, a curve K on P is transformed by B into a curve K' on P which touches K. This follows from the fact that P is a fixed point on *every* cubic of the pencil C_λ. On the other hand p_9 is a fixed point on but one cubic C_9 of the pencil C_λ and arises from the coincidence at p_9 in the direction of the tangent T_9 to C_9 at p_9 of a copair x, y of B. Hence this is one fixed direction on p_9, and another is the direction T_{Np_9} of N at p_9, *i. e.*, the direction to a neighboring fixed point. Any curve K on p_9 is transformed by B into a curve K' on p_9 such that the tangents to K and K' at p_9 are harmonic to T_9 and T_{Np_9}.

Every point x of the plane is a double point of at least one sextic of the web, namely of the squared cubic, C_1^2, on it. If x is a double point of a second sextic $\bar{\Sigma}$, and therefore of a pencil, then the net determined by C_1^2, C_1C_2, and $\bar{\Sigma}$ on x have their remaining intersection y at x, which may be at P if $\bar{\Sigma}$ is $C_\lambda^2(\lambda \neq 1)$, but otherwise is a point p_9 on N. Conversely the net of sextics on p_9 being fixed curves have as a common direction that of T_9 which belongs to the coincident pair, and therefore a pencil of the net will have a node at p_9 with nodal tangents harmonic to T_9 and to T_{Np_9}. The pencil contains one cuspidal sextic with tangent T_{Np_9} and one squared cubic C_9^2 with tangent T_9. Hence, disregarding nodes and cusps due to the sextics C_λ^2, and disregarding also the point P, we see that N is the locus of nodes of sextics of the web (13); or also the locus of cusps of sextics of the web; or as an envelope is the locus of cusp tangents; or finally is that 9-ic with triple points at $p_1, \ldots, p_8$ and on $D_1, \ldots, D_{12}$. For a double point of a cubic C_λ is projected into itself from a point of C_λ. An equation of N is the Jacobian, $J(C_1, C_2, \Sigma) = 0$.

The curve N is of genus 4 and its canonical series g_6^3 is cut out by the web of adjoints (13). The series cut out by the pencil C_λ, a g_3^1, has for residue with respect to g_6^3 the same g_3^1. Thus N differs from the general curve of genus 4 in that the two series, g_3^1, cut out on the norm curve by the two sets of generators of the quadric on the norm curve have coincided, *i. e.*, its canonical adjoints (13) map N into a space sextic cut out on a quadric cone by a cubic surface. Since the quadric is a cone, N has but eight moduli, the absolute constants of $p_1, \ldots, p_8$. A tangent plane of the quadric cone does not count as a tritangent plane of the sextic since it is rather a reunion of a set of g_3^1 and a set of $g_3^{1'}$. The 120 tritangent planes arise from the 120 degenerate sextics, $\binom{8}{1}$ of type $f_0(1) \cdot S_{p_1}$, $\binom{8}{2}$ of type $f_1(12) \cdot f_5(123^2 \ldots 8^2)$, $\binom{8}{5}$ of type $f_2(12345) f_4(123456^2 7^2 8^2)$, and $\binom{8}{6}$ of type $f_3(1^2 234567) \cdot f_3(2345678^2)$. Since a g_n^r has $(r+1)(n+rp-r)$ $(r+1)$-fold points, g_3^1 has twelve double points which are at $D_1, \ldots, D_{12}$. If p_9 is a general point on N there is as we have seen, a pencil of sextics with a node at p_9. This pencil cuts N in a g_4^1 with fourteen double points. Two of these double points arise from the two further intersections of the squared cubic C_9^2 on p_9. The remaining twelve are points p_{10} cut out by sextics with a node at p_{10} since all sextics on p_{10} with a simple point at p_{10} touch the cubic C_{10} at p_{10} and not N. Hence in a pencil of sextics with nodes at $p_1, \ldots, p_9$ there are precisely twelve rational sextics. In part this conclusion could be drawn as follows: If p_{10} is the 10-th node of a sextic with nodes at $p_1, \ldots, p_9$ then p_{10} lies both on N and on the 9-ic N'

formed like N with triple points at $p_1, \ldots, p_7, p_9$. Then N and N' meet in 7×9 points at $p_1, \ldots, p_7$ and in 2×3 points at p_8, p_9, whence p_{10} is one of the twelve remaining intersections. Thus there are *at most* twelve positions of p_{10}. It appears therefore that the three conditions that N be on p_9 and p_{10} and that N' be on p_{10} are necessary and sufficient conditions that P_{10}^2 be the nodes of a rational sextic.

The relation between p_9 and p_{10} gives rise to a symmetrical (12, 12) correspondence, T, on N. The valence of T is 3. For if C_3 is a set of the g_3^1, and C_2 the residue of that set on p_9, if K is a canonical set in g_6^2, and G a set of the g_4^1 considered above, and if S_{12} is the set of twelve positions of p_{10} when p_9 is given, then $S_{12}+C_2$ is the set of fourteen double points of the g_4^1. Hence $K+2G\equiv S_{12}+C_2$,* where now the equivalence refers to point groups on N. But $G+2p_9\equiv K$, and $C_2+p_9\equiv C_3$, and $2C_3\equiv K$ whence $S_{12}+3p_9\equiv 2K+C_3$. Hence if p_9' is any other point on N and S_{12}' its set of twelve additional nodes $S_{12}+3p_9\equiv S_{12}'+3p_9'$, or T has the valence $\gamma=3$. Then according to the well-known formula $\alpha+\beta+2p\gamma$, T has $12+12+24=48$ coincidences. These arise from those positions of p_9 where a rational sextic of the web has a tacnode, but also from the twelve points $D_1, \ldots, D_{12}$. For if C_λ has a node at D on N, then C_λ^2 meets N four times at D. Of this $4D$, the set $2D$ is eliminated in forming g_4^1, but $2D$ is left and D is a double point of g_4^1. Thus D belongs to the set S_{12} which corresponds to D in T and is therefore a coincidence. Hence

(14) *There are thirty-six sextics with eight given nodes which have an additional tacnode.*

Thus a sextic with a tacnode has only eight absolute constants. Miss Hudson's theorem that any rational sextic S for which a discriminant condition vanishes can be transformed into a sextic with a tacnode shows that S could have only eight absolute constants. For the tacnodal sextic can be transformed back into S by a series of quadratic involutions each with F-points and one fixed point at nodes of the sextic, and by a subsequent projectivity—a process which can introduce no new absolute constants.

The discriminant conditions furnish irrational invariants of the general sextic S. Symmetric combinations of those which lie within one of the five types of Theorem (10) furnish rational projective invariants of S. Symmetric combinations of the whole set of 496 furnish invariants of S under Cremona transformation of S into S'.

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* Severi, "Lezioni di Geometria Algebrica," p. 160.

§ 3. *The Group $\overline{G}_{10,2}$ of S.*

Since $\overline{G}_{10,2}$ is the group of all Cremona transformations which transform S into itself, the elements of $\overline{G}_{10,2}$ will either leave every point on S unaltered or transform the points of S among themselves according to a transformation on the parameter t of S of the form

$$t'=\frac{at+b}{ct+d}. \tag{15}$$

The group $\gamma_{10,2}$ of transformations (15) thus induced by $\overline{G}_{10,2}$ upon S will be simply isomorphic with $\overline{G}_{10,2}$ if the group $\overline{\overline{G}}_{10,2}$ of Cremona transformations for which every point of S is fixed is merely the identity. Otherwise $\gamma_{10,2}$ is the factor group of $\overline{\overline{G}}_{10,2}$ under $\overline{G}_{10,2}$.

The $\overline{G}_{10,2}$ is generated by the conjugates of the Bertini involution under $G_{10,2}$. If B is the involution with F-points at the nodes $p_1, \ldots, p_8$ of S, then we have just seen that B leaves the points p_9 and p_{10} unaltered and interchanges the two branches of S at each of these nodes. Hence if t_9, t_9' and t_{10}, t_{10}' are the pairs of nodal parameters, the transformation (15) induced by B interchanges the parameters in each pair and is the involution whose fixed points are the Jacobian of the nodal pairs. These fixed points are cut out on S by the curve N outside of P_{10}^2.

Two f-curves may meet at an F-point say p_i in P_{10}^2, but ordinarily they pass through p_i with different tangents, *i. e.*, they have at p_i different points in common with the f-curve, $f_0(i)$, which is made up of directions at p_i. We say then they have no *proper* intersection at p_i. Two f-curves may be selected so that they have any number of proper intersections. For as the order of the transformations of $G_{10,2}$ increases, the multiplicity of the f-curves of the transformations at F-points also increases so that the number of proper intersections of these f-curves and $f_0(i)$ increases without limit. Any two f-curves without proper intersections are conjugate under $G_{10,2}$. For the first can be transformed into $f_0(10)$ by an operation of $G_{10,2}$ which at the same time transforms the second into an f-curve on P_9^2; and this finally by an operation of $G_{9,2}$ which $f_0(10)$ unaltered can be transformed into $f_0(9)$. Also since every f-curve has precisely two proper intersections with S we have the theorem:

(16) *The group $\gamma_{10,2}$ of transformations* (15) *on S is generated by a conjugate set of involutions each determined by a pair of fixed points which is the Jacobian of the pairs of proper intersections with S of any two f-curves which have no proper intersections with each other.*

One may show in the same way that if ten f-curves are such that no two have proper intersections at P_{10}^2 they define a Cremona transformation of $G_{10,2}$. In fact the signatures of the f-curves furnish the columns of the matrix of L in (1).

If we transform the involution B by L the f-curves $f_0(9)$ and $f_0(10)$ become $f_{r_9}(1^{\alpha_{19}} \ldots . 10^{\alpha_{109}})$ and $f_{r_{10}}(1^{\alpha_{110}} \ldots . 10^{\alpha_{1010}})$. It merely requires a multiplication of three determinants to form the transform $L^{-1}BL$, and after evident reductions we find that the transformed form has coefficients

$$(17) \quad \begin{cases} m' = 17 + 12(r_9 + r_{10}) + 4r_9 r_{10}, \\ r_i' = s_i' = 6 + 2(r_9 + r_{10}) + 6(\alpha_{i9} + \alpha_{i10}) + 2(r_9\alpha_{i10} + r_{10}\alpha_{i9}), \\ (j \neq i)\,\alpha_{ji}' = 2 + 2(\alpha_{j9} + \alpha_{j10} + \alpha_{i9} + \alpha_{i10}) + 2(\alpha_{i9}\alpha_{j10} + \alpha_{j9}\alpha_{i10}), \\ \alpha_{ii} = 3 + 4(\alpha_{i9} + \alpha_{i10}) + 4\alpha_{i9}\alpha_{i10}. \end{cases}$$

(18) *If* $f_{r_9}(1^{\alpha_{19}} \ldots . 10^{\alpha_{109}})$ *and* $f_{r_{10}}(1^{\alpha_{110}} \ldots . 10^{\alpha_{1010}})$ *are two f-curves without proper intersections the conjugate of the Bertini involution determined as in* (16) *by the two when regarded as an element L of* $g_{10,2}$ *has the coefficients* (17).

The question as to whether $\overline{\overline{G}}_{10,2}$ contains elements other than the identity is related to the question as to whether the two proper intersections of distinct f-curves with S can coincide. For if $C \neq 1$ is an element of $\overline{\overline{G}}_{10,2}$ and leaves every point of S unaltered, it leaves the two directions of S at p_i unaltered, whence the f-curve which corresponds to p_i under C must pass through p_i with these two directions (and in general others). Thus this f-curve and $f_0(p_i)$ have the same pair of proper intersections with S. I am inclined to think that distinct f-curves meet S in distinct pairs, but have no proof that this is true.

PART II.

THE TEN NODES OF THE SYMMETROID.

§ 4. *The Dilation of a Regular Cremona Group.*

A *regular* Cremona transformation in S_k is by definition (P. S. II, § 4) any product of involutions of the type $y_i'y_i = C_i$ $(i = 1, 2, \ldots, k+1)$ where the products are formed with the $(k+1)$ F-points within a given point set as described in the introduction. The regular group $G_{n,k}$ attached to the point set P_n^k, transforms spreads of order x_0 and multiplicities $x_1, \ldots, x_n$ at P_n^k

according to the group $g_{n,k}$ of linear transformations L with coefficients (P. S. II, § 5 (23))

$$
(19) \qquad \begin{pmatrix}
(k-1)\mu+1-\rho_1 & -\rho_2 & \vdots & -\rho_n \\
(k-1)\sigma_1 & -\alpha_{11}-\alpha_{12} & \vdots & -\alpha_{1n} \\
(k-1)\sigma_2 & -\alpha_{21}-\alpha_{22} & \vdots & -\alpha_{2n} \\
\cdots\cdots & \cdots\cdots & \vdots & \cdots \\
(k-1)\sigma_n & -\alpha_{n1}-\alpha_{n2} & \vdots & -\alpha_{nn}
\end{pmatrix}.
$$

This group $g_{n,k}$ is generated by the permutation $g_{n!}$ of the n variables and the involution $A_{1,2,\ldots\ldots,k+1}$ whose coefficients are (P. S. II, § 5)

$$
(20) \qquad \begin{pmatrix}
k & -1 & -1 & \vdots & -1 & 0 & \vdots \\
k-1 & 0 & -1 & \vdots & -1 & 0 & \vdots \\
k-1 & -1 & 0 & \vdots & -1 & 0 & \vdots \\
\cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \vdots \\
k-1 & -1 & -1 & \vdots & 0 & 0 & \vdots \\
0 & 0 & 0 & \vdots & 0 & 1 & \vdots \\
\cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \vdots
\end{pmatrix}.
$$

Suppose then that the general element of $G_{n,k}$ has been obtained by forming a proper sequence Π of the products from $g_{n!}$ and $A_{1,\ldots\ldots,k+1}$. Consider a set of $n+l$ points in an S_{k+l}, *i. e.*, a set P_{n+l}^{k+l}. In this space separate a set of l of the points P_{n+l}^{k+l} (call these for the moment the *fixed* F-points) and order the remaining n points of P_{n+l}^{k+l} with respect to the points of P_n^k. Then in S_{k+l} form a product Π' of elements from $g_{(n+l)!}$ and $A_{1,\ldots\ldots,l,\,l+1,\ldots\ldots,l+k+1}$ in such a way that the last n points of P_{n+l}^{k+l} are permuted like the n points of P_n^k under $g_{n!}$, the first l remaining fixed. This requires that always in using an element A the first l of its F-points shall fall at the first l points of the set P_{n+l}^{k+l}. We shall then say that the element Π' of $G_{n+l,k+l}$ is the *element* Π *of* $G_{n,k}$ *dilated into* S_{k+l}. The element of $g_{n+l,k+l}$ which corresponds to the element Π' dilated from (19) has coefficients

$$
(21) \qquad \begin{pmatrix}
(k+l-1)\mu+1 & -\mu & -\mu & \vdots & -\mu & -\rho_1 & \cdots & -\rho_n \\
(k+l-1)\mu & -\mu+1 & -\mu & \vdots & -\mu & -\rho_1 & \cdots & -\rho_n \\
(k+l-1)\mu & -\mu & -\mu+1 & \vdots & -\mu & -\rho_1 & \cdots & -\rho_n \\
\cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots & \cdots \\
(k+l-1)\mu & -\mu & -\mu & \vdots & -\mu+1 & -\rho_1 & \cdots & -\rho_n \\
(k+l-1)\sigma_1 & -\sigma_1 & -\sigma_1 & \vdots & -\sigma_1 & -\alpha_{11} & \cdots & -\alpha_{1n} \\
\cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots & \cdots \\
(k+l-1)\sigma_n & -\sigma_n & -\sigma_n & \vdots & -\sigma_n & -\alpha_{n1} & \cdots & -\alpha_{nn}
\end{pmatrix}.
$$

In order to prove this we have only to show that the general element (19) multiplied by $A_{1,\ldots\ldots,k+1}$ when dilated according to the rule which is evident in (21) is the same as the dilated element (21) multiplied by $A_{1,\ldots\ldots,l+1,\ldots\ldots,l+k+1}$. We shall omit the verification which depends merely on determinant multiplication. Hence

(22) *The regular Cremona group attached to a set P_n^k in S_k when dilated into S_{k+l} furnishes a subgroup of the regular Cremona group in S_{k+l} attached to the set P_{n+l}^{k+l} which is simply isomorphic with the original group. The dilated group permutes the S_l's in S_{k+l} upon the l fixed F-points just as the original group permutes the points of S_k.*

In fact if the S_l's be cut by an S_k, which does not cut their common S_{l-1}, the original group appears in this S_k.

The following extension of P. S. II, § 4 (17) is now evident.

(23) *The group $G_{n,k}$ contains subgroups simply isomorphic with $G_{n',k'}$ whenever $n' \lesssim n$ and $k' \lesssim k$.*

We shall have occasion to use the dilations into S_3 of the Bertini involution, and of the Geiser involution in S_2 with triple F-points at $p_2, \ldots, p_8$. The matrices of these dilated transformations are, respectively,

$$(24) \quad \begin{pmatrix} 33 & -16 & -6 & -6 & \ldots & -6 \\ 32 & -15 & -6 & -6 & \ldots & -6 \\ 12 & -6 & 3 & -2 & \ldots & -2 \\ 12 & -6 & -2 & 3 & \ldots & -2 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 12 & -6 & -2 & -2 & \ldots & 3 \end{pmatrix}, \quad \begin{pmatrix} 15 & -7 & -3 & -3 & \ldots & -3 \\ 14 & -6 & -3 & -3 & \ldots & -3 \\ 6 & -3 & -2 & -1 & \ldots & -1 \\ 6 & -3 & -1 & -2 & \ldots & -1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 6 & -3 & -1 & -1 & \ldots & -2 \end{pmatrix}.$$

§ 5. *The Transforms of the Symmetroid by Regular Cremona Transformation.*

The symmetroid Σ is the quartic surface obtained by equating to zero a symmetric determinant of order 4 whose elements are linear forms. The ten points at which the first minors all vanish form the set P_{10}^3 of nodes of Σ. The enveloping cone of Σ from one of the nodes breaks up into two cones of the third order which meet in the nine lines to the other nodes. If this property appears at one node of a ten-nodal quartic surface, the surface is a symmetroid.

Let us call a set of eight points in space $p_1, \ldots, p_8$ a ***half-period set*** if on the elliptic quartic through the eight, the parameters satisfy the condition $u_1 + \ldots + u_8 \equiv \omega/2$, where $\omega/2$ is not the zero half-period when $v_1 + \ldots + v_4 \equiv 0$ is the coplanar condition. Let us further call $8+k$ points a ***half-period set*** if every set of eight in the set of $8+k$ points is itself a half-period set. Then a further property of Σ is that its set of nodes P_{10}^3 is a half-period set.*

From the property of the enveloping cone there follows:

(25) *If nine nodes of a symmetroid are given, the tenth is uniquely determined.*

(26) *A symmetroid is transformed by regular Cremona transformations with $\rho \gtrless 10$ F-points at P_{10}^3 into a symmetroid Σ' whose nodes $P_{10}^{3\prime}$ are congruent to P_{10}^3.*

For first if $p_1, \ldots, p_9$ are given, the line $\overline{p_1p_{10}}$ is determined as the 9-th base line of a pencil of cubic cones on the eight lines $\overline{p_1p_2}, \ldots, \overline{p_1p_{10}}$. Similarly the line $\overline{p_2p_{10}}$ is determined and thereby also the node p_{10}. Secondly a cubic transformation A_{1234} of the type $x_i'x_i = C_i$ $(i=1, \ldots, 4)$ with F-points at $p_1, \ldots, p_4$ transforms Σ into a quartic surface Σ' with nodes at a congruent set Q_{10}^4. Now A_{1234} is the dilation of a ternary quadratic transformation A_{234} which sends nine base points of a pencil of cubics on $p_2p_3p_4$ into a congruent set with a similar base point property whence A_{1234} has the same effect on the nine base lines through p_1, and Σ' is also a symmetroid. Moreover, any regular transformation of the sort described in (26) is a product of such cubic transformations.

It is our primary purpose to show that Σ can be transformed by such regular transformation into only a finite number of projectively distinct symmetroids, or since

(27) *There is but one symmetroid with given nodes*, that from the set P_{10}^3 of nodes of Σ only a finite number of projectively distinct congruent sets Q_{10}^3 can be derived.

In general there is an infinite number of sets Q_{10}^3 congruent to but projectively distinct from P_{10}^3 (P. S. II (14), (18)), and these arise from P_{10}^3 by the operations of the group $G_{10,3}$. If for the set P_{10}^3 of nodes of Σ this number is finite, there must be a subgroup $\overline{G}_{10,3}$ of $G_{10,3}$, which transforms Σ into itself, of finite index under $G_{10,3}$. We shall find that an important subgroup

* Cayley, *Coll. Math. Pap.*, Vol. VII, p. 304; Vol. VIII, p. 25.

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$G(2)$ of $\bar{G}_{10,3}$ is generated by the conjugates under $G_{10,3}$ of two types of involutions, namely, the "Kantor involution"* and the dilated Bertini involution. In $g_{10,3}$ there are the isomorphic subgroups $\bar{g}_{10,3}$ and $g(2)$.

The Kantor involution K is that cut out on elliptic quartic curves on $p_1, \ldots, p_7$ by quartic surfaces with nodes at $p_1, \ldots, p_7$. It has for fixed points the 8-th node of such surfaces; and these are the 8-th base point P of the net of quadrics on $p_1, \ldots, p_7$—an isolated fixed point with fixed directions—and the locus of the point p_8 which forms with $p_1, \ldots, p_7$ a half-period set—the Cayley dianome sextic surface. Hence p_8, p_9, p_{10}, the further nodes of Σ, are fixed points of K, and Σ is unaltered by K. This involution is the analog in space of the Bertini involution in the plane.

In order to show that the dilated Bertini involution also leaves Σ unaltered, two lemmas are useful.

(28) *The dilation from p_1 of the Geiser involution with F-points $p_2, \ldots, p_8$ in S_2 is, in S_3, the transformation* (24) *whose two sets of F-points $p_1, \ldots, p_8$ and $q_1, \ldots, q_8$ are projective only when they are half-period sets. If the two sets are thus restricted and coincide in the identical order, the dilated transformation is involutory.*

For the dilation of this involution is found listed in P. S. II, p. 376, as $C(\nu)(\nu=-1)$. It is shown there that $C(-1)C(0)=D_1$ or $C(-1)=D_1C(0)$ where $C(0)$ is the Kantor involution determined by $p_2, \ldots, p_8$. It is clear from the parametric equations of D_1 (*loc. cit.*) that its two sets of F-points are projective if they are half-period sets. In this case p_1 is a fixed point of $C(0)$ and the two sets of F-points of $C(-1)$ are projective. If for $C(-1)$, P_8^3 and Q_8^3 coincide then $p_1, \ldots, p_8$ are ordinary points of $[C(-1)]^2$ and $C(-1)$ is involutory.

(29) *The dilation from p_1 of the Bertini involution with F-points $p_2, \ldots, p_9$ in S_2 is, in S_3, the transformation* (24) *whose two sets of F-points $p_1, \ldots, p_9$; $q_1, \ldots, q_9$ are projective only when these sets are half-period sets. If they are thus restricted and coincide in the identical order, the dilated transformation is involutory.*

For in P. S. II, p. 353, the Bertini involution (E_{17}) is expressed as a product of three Geiser involutions (D_8) and from the projectivity of the two

* The Kantor involution appears first in two papers of S. Kantor, "Theorie der periodischen cubischen Transformationen im R_3," this Journal, Vol. XIX (1897), p. 1; and "Theorie der Transformationen im R_3," *Acta Mathematica*, Vol. XXI (1897), p. 1, both of which deal with regular transformations in S_3. A development of the properties of the involution is given by J. R. Conner, "Correspondences Determined by the Bitangents of a Quartic," this Journal, Vol. XXXVIII (1916), p. 155.

sets of seven F-points of the factors, the projectivity of any two corresponding sets of six points from the two sets of eight F-points of the product was derived. Because of the isomorphism between elements in S_2 and their dilations in S_3 the dilated Bertini involution can be expressed as a similar product of three dilated Geiser involutions. Hence by virtue of (28) we can conclude that the pair p_1q_1 and any six further pairs of F-points are projective when $p_1, \ldots, p_9$ is a half-period set. Hence the two sets of F-points of the dilated Bertini involution are projective when one is a half-period set, and if the two sets coincide the square of the transformation is the identity.

We can now proceed with Σ very much as with the sextic S and will state first the analog of Theorem (4), § 1.

(30) *The group, $G(2)$, of regular transformations in S_3, generated by the conjugates of the Kantor and dilated Bertini involutions under $G_{10,3}$, is an invariant subgroup of $\overline{G}_{10,3}$ which leaves Σ unaltered. The isomorphic group, $g(2)$, is that subgroup of $g_{10,3}$ which is congruent to the identity modulo* 2.

Indeed we have already remarked that K leaves Σ unaltered. There follows directly from (29), (25) and (27) that B has the same property. That the conjugates of K, B under $G_{10,3}$ have this property is proved as for the sextic. In order to prove that the involutions generate $G(2)$ we indicate as before by the symbol $\equiv$ equivalence under $G(2)$.

If P_{10}^3 is the set of nodes of Σ it determines a sequence of f-surfaces, the conjugates of the ∞^2 directions about $p_1, \ldots, p_{10}$ under the operations of $G_{10,3}$. We construct like the Table (5) for the sextic the Table (31), discarding as new types those f-surfaces which are equivalent under $G(2)$ to types found earlier. Non-equivalent new types are starred as they occur.

In order to prove the equivalences listed in the table we shall prove first that the following list is valid.

(12⁰) $$f_3(1^2 2 5^3 678910) \equiv f_3(1^2 2^3 5678910).$$

(13⁰) $$f_3(12^3 5^2 678910) \equiv f_3(12^3 3^2 678910).$$

(14⁰) $$f_2(12^2 3456) \equiv f_2(134567^2).$$

(15⁰) $$f_4(1^3 2^4 345678910) \equiv f_4(12^4 3^3 45678910).$$

We begin with the equivalences obtained from K and B,

$$f_0(i) \equiv f_4(i^3 j_1^2 \ldots j_6^2) \equiv f_6(i^3 j_1^6 j_2^2 \ldots j_8^2),$$

and transform them successively by $A_{ij_1j_2j_3}$, $A_{ij_1j_4j_5}$, $A_{ij_1j_6j_7}$, and $A_{ij_1j_8j_9}$ getting

$$
\begin{aligned}
&f_1(j_1j_2j_3) &&\equiv f_3(i^2j_1j_2j_3j_4^2j_5^2j_6^2) &&\equiv f_5(i^2j_1^5j_2j_3j_4^2\ldots\ldots j_8^2),\\
&f_2(ij_1^2j_2j_3j_4j_5) &&\equiv f_2(ij_2j_3j_4j_5j_6^2) &&\equiv f_4(ij_1^4j_2j_3j_4j_5j_6^2j_7^2j_8^2),\\
&f_3(i^2j_1^3j_2\ldots\ldots j_7) &&\equiv f_3(i^2j_1\ldots\ldots j_5j_6^3j_7) &&\equiv f_3(j_1^3j_2\ldots\ldots j_7j_8^2),\\
&f_4(i^3j_1^4j_2\ldots\ldots j_9) &&\equiv f_6(i^5j_1^4j_2\ldots\ldots j_5j_6^3j_7j_8^3j_9^3) &&\equiv f_4(ij_1^4j_2\ldots\ldots j_8j_9^3).
\end{aligned}
$$

(31)

Type	operated upon by	becomes	which is	
$f_0(1)$	A_{1234}	$f_1(234)$*		
	A_{2345}	$f_0(1)$		
$f_1(234)$	A_{1234}	$f_0(1)$		
	A_{1235}	$f_1(234)$		
	A_{1256}	$f_2(12^23456)$*		
	A_{1567}	$f_3(1^22345^26^27^2)$	$\equiv f_1(234)$	(1^0)
$f_2(12^23456)$	A_{1234}	$f_1(256)$		
	A_{3456}	$f_2(12^23456)$		
	A_{1237}	$f_2(12^23456)$		
	A_{4567}	$f_3(12^234^25^26^27)$	$\equiv f_1(137)$	(1^0)
	A_{1278}	$f_3(1^22^3345678)$*		
	A_{5678}	$f_4(12^2345^36^37^28^2)$	$\equiv f_2(12^23456)$	(2^0)
	A_{2789}	$f_4(12^434567^28^29^2)$	$\equiv f_2(12^23456)$	(3^0)
	A_{1789}	$f_5(1^42^234567^38^39^3)$	$\equiv f_3(1^234567^389)$	(4^0)
	A_{78910}	$f_6(12^234567^48^49^410^4)$	$\equiv f_2(12^23456)$	(5^0)
$f_3(1^22^3345678)$	A_{1234}	$f_2(12^25678)$		
	A_{2345}	$f_3(1^22^3345678)$		
	A_{1345}	$f_4(1^32^33^24^25^2678)$	$\equiv f_2(123^2678)$	(2^0)
	A_{3456}	$f_5(1^22^33^34^35^36^378)$	$\equiv f_3(1^22^3345678)$	(6^0)
	A_{1289}	$f_3(1^22^3345678)$		
	A_{2789}	$f_4(1^22^434567^28^29)$	$\equiv f_2(1^234569)$	(3^0)
	A_{1789}	$f_5(1^42^334567^38^39^2)$	$\equiv f_3(1^22^3345678)$	(4^0)
	A_{6789}	$f_6(1^22^33456^47^48^49^3)$	$\equiv f_2(1^223459)$	(7^0)
	A_{12910}	$f_4(1^32^43\ldots10)$*		
	A_{28910}	$f_5(1^22^53\ldots78^39^210^2)$	$\equiv f_3(1^22^3345678)$	(8^0)
	A_{18910}	$f_6(1^52^33\ldots78^49^310^3)$	$\equiv f_4(1^32\ldots78^4910)$	(9^0)
	A_{78910}	$f_7(1^22^334567^58^59^410^4)$	$\equiv f_3(1^22^3345678)$	(10^0)
$f_4(1^32^43\ldots\ldots10)$	A_{1234}	$f_3(1^22^35678910)$		
	A_{2345}	$f_5(1^32^53^24^25^2678910)$	$\equiv f_3(12^33^2678910)$	(8^0)
	A_{1345}	$f_6(1^52^43^34^35^3678910)$	$\equiv f_4(1^32^43\ldots10)$	(9^0)
	A_{3456}	$f_8(1^32^43^54^55^56^578910)$	$\equiv f_4(1^32^43\ldots10)$	(11^0)

In these transforms we find (12^0), (13^0), (14^0), (15^0) as well as (1^0), (2^0) and (9^0). Also (2^0) is transformed by A_{1234} into (6^0), whence (6^0) is valid. Since (8^0) is transformed by A_{1234} into (13^0), (8^0) also is valid. Again (7^0) is

transformed by A_{1267} and the use of (13^0) into (4^0), and (4^0) by A_{1789} into (14^0). Also (3^0) is transformed by A_{1234} into one proved above. The equivalence (11^0) is transformed by A_{1234} into (10^0), and (10^0) by A_{1278} into (5^0). Finally, by using (14^0) we write (5^0) as $f_6(12^2 34567^4 8^4 9^4 10^4) \equiv f_2(134567^2)$ and this is transformed by A_{1789} and the use of (13^0) into (4^0). According to the equivalences derived above from the conjugates of K and B we find that all f-surfaces whose signatures are congruent modulo 2 are equivalent under $G(2)$ which completes the proof of (30).

The factor group of $g(2)$ under $g_{10,3}$ is the group $g_{10,3}^{(2)}$ of transformations L reduced modulo 2. According to the table (T. M. Groups, p. 337, $\varkappa=3$, $\nu=2$) this group has the order $\mu=2^9.2^{16}(2^8-1)(2^6-1)(2^4-1)(2^2-1)$. Also μ is the index of $G(2)$ under $G_{10,3}$. There may be elements in $G_{10,3}$ other than those in $G(2)$ which leave Σ unaltered. Consider the μ transforms of Σ under $G_{10,3}$. In these transforms we find that the f-surface $f_0(10)$ is transformed into 2^9 conjugates not equivalent under $G(2)$. These are of the five types listed in the first column of Table (31), there being $\binom{10}{1}$, $\binom{10}{3}$, $\binom{10}{5}$, $\binom{10}{7}$, $\binom{10}{9}$ of the respective types. Hence, under the operations of $G_{10,3}$ for which p_{10} is an ordinary point, we would find only $\mu'=\mu/2^9$ transforms of Σ. Under the latter operations the f-surface $f_0(9)$ is transformed into 2^8-1 conjugates not equivalent under $G(2)$, namely the $\binom{9}{1}$, $\binom{9}{3}$, $\binom{9}{5}$, $\binom{9}{7}$ of the first four types just mentioned. Hence under the operations of $G_{10,3}$ for which both p_9 and p_{10} are ordinary points, we would get only $\mu''=\mu'/(2^8-1)=2^{16}(2^6-1)(2^4-1)(2^2-1)$ transforms of Σ, and these recur in sets of $8!$ obtained by permutation of $p_1, \ldots, p_8$. Thus we should get only $\mu''/8!=2.2^6.36$ projectively distinct sets of nodes $p_1, \ldots, p_8$. On the other hand we have proved (P. S. II, p. 377 (46)) that when P_8^3 is a half-period set, there are only $2^6.36$ projectively distinct sets congruent in some order to P_8^3.

This indicates the existence of Cremona transformations not in $G(2)$ which have their F-points at $p_1, \ldots, p_8$ alone and which transform Σ into itself. Indeed

(32) *The dilated Geiser involution with F-points at the nodes* $p_1, \ldots, p_8$ *of* Σ *transforms* Σ *into itself and interchanges the nodes* p_9 *and* p_{10}.

For let us first recall with Rohn * that when the first seven nodes of Σ are given, the other three lie on Cayley's dianome sextic surface with triple points at $p_1, \ldots, p_7$. Having chosen p_8 on this surface, the other two nodes lie on

* K. Rohn, "Die Flächen vierter Ordnung," etc., Jablonowski'schen Preisschrift, Leipzig (1886), §§ 9, 10, 11.

Cayley's dianodal curve of order 18 with planar triple points at the eight nodes. As Rohn remarks, the ninth being chosen, the tenth is uniquely determined if the quartic is to be a symmetroid. This follows immediately from (25). Thus there is on the dianodal curve an involution of pairs of nodes of symmetroids. Now this involution is effected by the Geiser involution dilated from p_1 (and therefore also that the Geiser involution dilated from any other of the eight nodes). For since the eight nodes are a half-period set, the dilated transformation is involutory (28) when its two sets of eight F-points coincide. Moreover, the dilated transformation is regular and transforms symmetroids into symmetroids (26) and therefore leaves the dianodal curve unaltered. If p_9, p_9' are a copair of the dilated involution on the curve, then from (22) the lines $\overline{p_1p_9}$, $\overline{p_1p_9'}$ form with $\overline{p_1p_2}, \ldots, \overline{p_1p_8}$ the base lines of a pencil of cubic cones. But this property is shared by the lines $\overline{p_1p_9}$ and $\overline{p_1p_{10}}$ when p_9, p_{10} are nodes of the same symmetroid. Hence p_9' is p_{10} and the theorem is proved.

Consider now the reduced group $g_{10,3}^{(2)}$ of $g_{10,3}$. The dilated Geiser involution reduced modulo 2 is

$$I_{12\ldots 8}I_{910} \text{ or } x_j'=x_j+(x_1+\ldots+x_8) \quad (j=0, 1, \ldots, 8), \quad x_9'=x_{10}, \quad x_{10}'=x_9$$

in the notation of T. M. Groups.* This is an element T (cf. p. 326, *loc. cit.*) which lies in the invariant g_{2^9} of $g_{10,3}^{(2)}$. If an element of $G_{10,3}$ leaves Σ unaltered, its conjugates have the same property whence those elements of $g_{10,3}^{(2)}$ conjugate to T under $g_{10,3}^{(2)}$, also correspond to elements of $G_{10,3}$ which leave Σ unaltered. Now the factor group of g_{2^9} under $g_{10,3}^{(2)}$ is the *simple* group $G_{NC}(p=4)$ of the odd and even thetas for $p=4$. Hence there are no further elements of $G_{10,3}$ which leave Σ unaltered since any such element reduced modulo 2 would furnish an invariant subgroup of $g_{10,3}^{(2)}$ larger than g_{2^9} whose factor group under $g_{10,3}^{(2)}$ would be the factor group under G_{NC} of an invariant subgroup of G_{NC} greater than the identity. But no such subgroup of G_{NC} exists. Hence the number $\bar{\mu}$ of transforms of Σ under $G_{10,3}$ is the order of G_{NC}, *i. e.*, $\bar{\mu}=2^{16}(2^8-1)(2^6-1)(2^4-1)(2^2-1)$ and allowing for the permutations of the nodes there are only $\bar{\mu}/10!=2^8.51$ projectively distinct Σ's. Hence

(33) *Under regular Cremona transformation a symmetroid* Σ *can be transformed into precisely* $2^8.51$ *projectively distinct* Σ'*s. The subgroup* $\bar{G}_{10,3}$ *of* $G_{10,3}$, *which leaves* Σ *unaltered is generated by the conjugates*

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* Cf. particularly the table, p. 337, for $\kappa=3$, $\nu=2$, and also (28) and (29) with references there given.

under $G_{10,3}$ of the dilated Geiser involution and the Kantor involution. The corresponding elements of $\bar{g}_{10,3}$ are characterized arithmetically by the fact that when reduced modulo 2 they yield either the identity or elements which transform the forms b_2, b_4 each into itself or into its paired form.† The invariant subgroup $G(2)$ of $G_{10,3}$ for which $g(2)$ is congruent to the identity modulo 2 is generated by the conjugates of the Kantor and dilated Bertini involutions, and has for factor group under $\bar{G}_{10,3}$ an abelian group of involutions of order 2^9. Under $G_{10,3}$ the conjugates of Σ are permuted according to the group of odd and even thetas for $p=4$, the particular types corresponding to the base configurations.‡*

We may note finally the behavior of the discriminant factors of the set P_{10}^3 of nodes of Σ. Due to the equivaleuces under $G(2)$ listed above we find that all of the discriminant conditions are equivalent to the following sets: $\binom{10}{2}$ of type $f_0(i_1 i_2)$, $\binom{10}{4}$ of type $f_1(i_1 i_2 i_3 i_4)$, $\binom{10}{6}$ of type $f_2(i_1^2 i_2 \ldots . i_7)$, and $\binom{10}{8}$ of type $f_3(i_1^3 i_2^2 i_3 \ldots . i_9)$, or $2(2^8-1)$ in all. But due to the equivalences under elements of $\bar{G}_{10,3}$ not in $G(2)$, these are paired into 2^8-1 pairs, $\binom{10}{2}$ of type $f_0(i_1, i_2)$, $f_3(i_1^2 i_3^3 i_4 \ldots . i_{10})$ and $\binom{10}{4}$ of type $f_1(i_1 i_2 i_3 i_4)$, $f_2(i_1^2 i_5 \ldots . i_{10})$. These two types of equivalence lead to the theorems

(34) *If two nodes of a symmetroid coincide, the cubic cone with vertex at any any one of the remaining nodes and on the ten nodes has a double generator on the double node.*

(35) *If four nodes of a symmetroid are in a plane there is a quadric cone with vertex at any one of the four nodes and on the remaining six nodes.*

When none of the discriminant conditions are satisfied they become irrational invariants of the symmetroid whose behavior under $G_{10,3}$ can be described thus:

(36) *Under regular Cremona transformation the 2^8-1 independent discriminant invariants of Σ are permuted like the points of an $S_{2p-1}(p=4)$ under the group of a null system in S_{2p-1}.*

This striking analogy with the 2^6-1 discriminant invariants of P_7^2 (or the ternary quartic for $p=3$; cf. P. S. II, § 8) is undoubtedly significant.

URBANA, ILLINOIS, *May* 15, 1919.

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* The dilated Bertini involution can be generated by dilated Geiser involutions.

† The forms b_2 are $x_{i_1}+x_{i_2}$, $x_{i_1}+\ldots .+x_{i_6}$, the forms b_4 are $x_{i_1}+\ldots .+x_{i_4}$ and $x_{i_1}+\ldots .+x_{i_8}$; paired forms taken together make up $x_{i_1}+\ldots .+x_{i_{10}}$ $(i_j=1, \ldots ., 10)$.

‡ For these configurations cf. a paper of the author on "The Finite Geometry of the Theta Functions," *Trans. Amer. Math. Soc.*, Vol. XIV (1913), p. 271.

Functions of Matrices.

By H. B. Phillips.

1. It is the purpose of the present paper to study the functions represented by polynomials or convergent series in a matrix or a finite number of matrices. As the work is concerned mainly with the roots of the matrices, the fundamental facts about the roots are first briefly developed.*

By a matrix of the n-th order is meant a square array of n^2 elements a_{ik}, $i, k=1, 1, \ldots, n$,

$$A=\begin{Vmatrix} a_{11} & a_{12} \ldots & a_{1n} \\ a_{21} & a_{22} \ldots & a_{2n} \\ \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} \ldots & a_{nn} \end{Vmatrix}=\|a_{ik}\|.$$

The matrices considered in this paper will all be of the same order.

The determinant

$$|A|=\begin{vmatrix} a_{11} & a_{12} \ldots & a_{1n} \\ a_{21} & a_{22} \ldots & a_{2n} \\ \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} \ldots & a_{nn} \end{vmatrix}=|a_{ik}|$$

is called the determinant of A. When this determinant is zero, A is called singular.

The sum of two matrices $A=\|a_{ik}\|$ and $B=\|b_{ik}\|$ is the matrix

$$A+B=\|a_{ik}+b_{ik}\|.$$

More generally, if λ and μ are numbers, $\lambda A+\mu B=\|\lambda a_{ik}+\mu b_{ik}\|$. A matrix is zero when and only when all its elements are zero.

The product AB of $A=\|a_{ik}\|$ and $B=\|b_{ik}\|$ is the matrix

$$AB=\|\alpha_{ik}\|, \quad \text{where} \quad \alpha_{ik}=\sum_{j=1}^{n} a_{ij}b_{jk}.$$

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* The general theory of matrices is given in Bocher's "Introduction to Higher Algebra." A very complete bibliography of literature before 1907 is given in James Byrnie Shaw's "Synopsis of Linear Associative Algebra," published by the Carnegie Institution of Washington.

The products AB and BA are not in general equal. If these products are equal, A and B are called commutative.

The products of three or more matrices are associative, that is,

$$(AB)C = A(BC) = ABC.$$

The determinant of a product of matrices is equal to the product of their determinants, for example, $|ABC| = |A| \cdot |B| \cdot |C|$.

The matrix

$$I = \begin{Vmatrix} 1 & 0 & 0 \ldots . 0 \\ 0 & 1 & 0 \ldots . 0 \\ \cdots & \cdots & \cdots \\ 0 & 0 & 0 \ldots . 1 \end{Vmatrix}$$

in which the elements a_{ii} are all unity and the others zero, is called the unit matrix. It is easy to see that $AI = IA = A$.

If the determinant of A is not zero, there is a matrix A^{-1} called the reciprocal of A, such that $AA^{-1} = A^{-1}A = I$. In this case, if $AB = CD$, $B = A^{-1}CD$, *i. e.*, we can divide by a non-singular matrix.

It is to be noticed that in both multiplication and division the operation can be performed on the right or on the left. B is multiplied on the left by A if the result is AB, and on the right by A if the result is BA. Similarly, B is divided on the left by A if the result is $A^{-1}B$ and on the right if the result is BA^{-1}.

2. *Identical Equations.*—The matrices $A, B, \ldots, P$ satisfy the equation

$$A + B + \ldots . + P = 0$$

if their sum is a matrix $\|\sigma_{ik}\|$ with elements σ_{ik} all zero. Cayley first observed that a matrix of the n-th order satisfies an algebraic equation of the n-th degree. This may be considered a consequence of the following theorem:

THEOREM I. *Let $A = \|a_{ik}\|$, $B = \|b_{ik}\|$, $\ldots$, $P = \|p_{ik}\|$ be a finite number of matrices of the n-th order and let $\sigma_{ik} = \lambda a_{ik} + \mu b_{ik} + \ldots . + \rho p_{ik}$, $\lambda, \mu, \ldots, \rho$ being numerical parameters. If $A', B', \ldots, P'$ are commutative with each other and satisfy the equation*

$$AA' + BB' + \ldots . + PP' = 0, \tag{1}$$

they will also satisfy the n-th degree equation

$$|a_{ik}A' + b_{ik}B' + \ldots . + p_{ik}P'| = 0, \tag{2}$$

obtained by replacing $\lambda, \mu, \ldots, \rho$ in $|\sigma_{ik}| = 0$ by the matrices $A', B', \ldots, P'$, respectively.

This theorem enables us to replace a homogeneous linear equation with matrix coefficients by one of the n-th degree with numerical coefficients. The latter equation will usually have solutions, however, that do not satisfy the former.

To prove the theorem, let E_{ik} be the matrix of the n-th order with all its elements zero except that in the i-th row and k-th column, which is unity. It is readily seen that

$$E_{ij}E_{jk}=E_{ik}, \quad E_{ij}E_{lk}=0, \quad l\neq j. \tag{3}$$

The matrix A can be written

$$A=\sum_{i,\,k=1}^{n} a_{ik}E_{ik},$$

and so (1) is equivalent to

$$\sum_{i,\,k=1}^{n} E_{ik}(a_{ik}A'+b_{ik}B'+\ldots\ldots+p_{ik}P')=0.$$

Multiplying this equation on the left by $E_{11}, E_{12}, \ldots\ldots, E_{1n}$, respectively, and using (3), we get

$$\begin{array}{l}
E_{11}(a_{11}A'+\ldots\ldots+p_{11}P')+E_{12}(a_{12}A'+\ldots\ldots+p_{12}P')+\ldots\ldots+E_{1n}(a_{1n}A'+\ldots\ldots+p_{1n}P')=0,\\
E_{11}(a_{21}A'+\ldots\ldots+p_{21}P')+E_{12}(a_{22}A'+\ldots\ldots+p_{22}P')+\ldots\ldots+E_{1n}(a_{2n}A'+\ldots\ldots+p_{2n}P')=0,\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
E_{11}(a_{n1}A'+\ldots\ldots+P_{n1}P')+E_{12}(a_{n2}A'+\ldots\ldots+p_{n2}P')+\ldots\ldots+E_{1n}(a_{nn}A'+\ldots\ldots+p_{nn}P')=0.
\end{array}$$

Since $A', B', \ldots\ldots, P'$ are commutative, we can eliminate $E_{12}, E_{13}, \ldots\ldots, E_{1n}$ by multiplying these equations (on the right) by the cofactors of the corresponding elements in the first column of the determinant

$$|a_{ik}A'+b_{ik}B'+\ldots\ldots+p_{ik}P'| \tag{4}$$

and adding. The result is

$$E_{11}|a_{ik}A'+b_{ik}B'+\ldots\ldots+p_{ik}P'|=0. \tag{5}$$

If we expand (4) and combine terms, the result is a matrix. Equation (5) shows that all the elements in the first row of that matrix are zero. For, the product of E_{11} and any matrix $\|\alpha_{ik}\|$ has as first row $\alpha_{11}, \alpha_{12}, \ldots\ldots, \alpha_{1n}$. If then $E_{11}\|\alpha_{ik}\|$ is zero, the elements $\alpha_{11}, \alpha_{12}, \ldots\ldots, \alpha_{1n}$ must all be zero. Similarly, we could eliminate $E_{11}, E_{13}, \ldots\ldots, E_{1n}$ and so get

$$E_{12}|a_{ik}A'+b_{ik}B'+\ldots\ldots+p_{ik}P'|=0,$$

showing that all the elements in the second row of the matrix represented by (4) are zero. By continuation of this argument we conclude finally that

$$|a_{ik}A'+b_{ik}B'+\ldots\ldots+p_{ik}P'|=0.$$

3. *Characteristic Equation.*—Let I be the unit matrix. Applied to the identity $AI-IA=0$, Theorem I gives

$$\begin{vmatrix} a_{11}I-A, & a_{12}I, & \dots\dots & a_{1n}I \\ a_{21}I, & a_{22}I-A, & \dots\dots & a_{2n}I \\ \dots & \dots & \dots & \dots \\ a_{n1}I, & a_{n2}I, & \dots\dots & a_{nn}I-A \end{vmatrix}=0.$$

This is an equation of the n-th degree in A called the *characteristic* equation of A. Arranged in descending powers of A (with a change of sign if n is odd), it takes the form

$$\phi(A)=A^n+\alpha_1A^{n-1}+\dots\dots+\alpha_nI=0. \tag{6}$$

Let $a_1, a_2, \dots\dots, a_n$ be the roots of the equation

$$\phi(r)=r^n+\alpha_1r^{n-1}+\dots\dots+\alpha_n=0.$$

Then $\phi(r)$ can be factored in the form

$$\phi(r)=(r-a_1)(r-a_2)\dots\dots(r-a_n).$$

Since this is an identity in r, the coefficients of each power of r on the two sides of the equation are equal. It will then still hold when r is replaced by A, and $a_1, \dots\dots, a_n$ by $a_1I, \dots\dots, a_nI$. Hence

$$\phi(A)=(A-a_1I)(A-a_2I)\dots\dots(A-a_nI).$$

The numbers $a_1, a_2, \dots\dots, a_n$ are called roots of the matrix A. These roots satisfy the equation

$$\phi(r)=\begin{vmatrix} a_{11}-r & a_{12} & \dots\dots a_{1n} \\ a_{21} & a_{22}-r & \dots\dots a_{2n} \\ \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots\dots a_{nn}-r \end{vmatrix}=0.$$

This expresses that the determinant $|A-rI|$ is zero. The roots of A are therefore the values of r for which the determinant $|A-rI|$ is zero.

A matrix of the n-th order may satisfy an equation of lower than the n-th degree. The equation of lowest degree satisfied by a given matrix will, however, be unique. For, if A satisfies two equations of the m-th degree, we can eliminate A^m from them and so find an equation of lower degree satisfied by A. Let

$$\psi(A)=A^m+c_1A^{m-1}+\dots\dots+c_mI=0$$

be the equation of lowest degree satisfied by A. This is sometimes called the reduced equation for A. It is clear that

$$\psi(r)=r^m+c_1r^{m-1}+\dots\dots+c_m$$

is a factor of $\phi(r)$. For, by division, we get

$$\phi(r)=Q(r)\psi(r)+R(r),$$

where $Q(r)$ is the quotient, and $R(r)$, of degree less than m, is the remainder. Since this is an identity in r, we can replace r by A and so get

$$\phi(A)=Q(A)\psi(A)+R(A).$$

But $\phi(A)$ and $\psi(A)$ are both zero. If then $R(r)$ were not identically zero, $R(A)=0$ would be an equation of lower than the m-th degree satisfied by A. Hence $R(r)=0$ and $\phi(r)=Q(r)\psi(r)$.

This shows that all the roots of the reduced equation are roots of the characteristic equation. Conversely, all the roots of the characteristic equation satisfy the reduced equation. For, since $\psi(r)$ is a polynomial, we can factor $\psi(r)-\psi(s)$ in the form

$$\psi(r)-\psi(s)=(r-s)P(r,s),$$

where $P(r,s)$ is a polynomial in r and s. Since this is an identity we can replace s by A and r by rI. Then, since $\psi(A)=0$,

$$\psi(r)I=(rI-A)P(rI,A).$$

Equating determinants of the two sides, we get

$$[\psi(r)]^n=|rI-A|\cdot|P(rI,A)|.$$

If now r is a root of A, $|rI-A|=0$, and so $\psi(r)=0$ which was to be proved.

4. *Associated Roots.*—THEOREM II. *If $A, B, \ldots, P$ are commutative matrices with roots $a_1, a_2, \ldots, a_n$; $b_1, b_2, \ldots, b_n$, etc., those roots can be arranged in sets $a_i, b_i, \ldots, p_i$, $i=1, 2, \ldots, n$ such that, if $f(a, b, \ldots, p)$ is any polynomial in $a, b, \ldots, p$, the roots of the matrix $f(A, B, \ldots, P)$ are*

$$f(a_i, b_i, \ldots, p_i), \quad i=1, 2, \ldots, n.$$

This theorem is due to Frobenius.* The following proof is somewhat more direct than the one given by him. Let $f_1(A, B, \ldots, P)$, $f_2(A, B, \ldots, P)$, etc., be polynomials and

$$f(A, B, \ldots, P)=\sum_{i=1}^{k}\lambda_i f_i(A, B, \ldots, P), \tag{7}$$

$\lambda_1, \lambda_2, \ldots, \lambda_k$ being arbitrary numbers. Since $A, B, \ldots, P$ have characteristic equations of the n-th degree, there are only a finite number of linearly independent polynomials in those matrices. We may consider $f_i(A, B, \ldots, P)$, $i=1, 2, \ldots, k$ as those polynomials, and so by a proper choice of $\lambda_1, \lambda_2, \ldots, \lambda_k$ make $f(A, B, \ldots, P)$ equal to any given polynomial in $A, B, \ldots, P$.

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* Sitzungsberichte Berliner Akademie (1896), p. 602.

By Taylor's theorem,

$$f(A, B, \ldots, P)-f(a_i, b_j, \ldots, p_k)I$$
$$=(A-a_iI)L+(B-b_jI)M+\ldots+(P-p_kI)R, \quad (8)$$

where $L, M, \ldots, R$ are polynomials in $A, B, \ldots, P, a_i, b_j$, etc. Form the product

$$\Pi[f(A, B, \ldots, P)-f(a_i, b_j, \ldots, p_k)I]$$

of all the differences (8) obtained by letting $i, j, \ldots, k$ vary independently from 1 to n. This product is zero. For when the product is expanded in terms of the quantities $(A-a_iI)$, $(B-b_jI)$, etc., each term of the result not containing $(B-b_jI), \ldots, (P-p_kI)$, $j, \ldots, k$ being definitely assigned, will contain all the factors $(A-a_iI)$, $i=1, 2, \ldots, n$. Hence each term contains one or more of the products

$$\begin{gathered}(A-a_1I)(A-a_2I)\ldots(A-a_nI),\\(B-b_1I)(B-b_2I)\ldots(B-b_nI),\\\ldots\ldots\ldots\ldots\ldots\ldots,\\(P-p_1I)(P-p_2I)\ldots(P-p_nI).\end{gathered}$$

These all vanish because the result of equating any one of them to zero is the characteristic equation of one of the matrices. Therefore

$$\Pi[f(A, B, \ldots, P)-f(a_i, b_j, \ldots, p_k)I]=0 \quad (9)$$

for all values of the parameters $\lambda_1, \lambda_2, \ldots, \lambda_k$.

Equation (9) is a polynomial equation in $f(A, B, \ldots, P)$. If the equation of least degree satisfied for all values of $\lambda_1, \lambda_2, \ldots, \lambda_k$ by that matrix is

$$\phi(x)=0, \quad (10)$$

$\phi(x)$ will be a divisor of

$$\Pi[x-f(a_i, b_j, \ldots, p_k)].$$

The roots of (10), *i. e.*, the roots of $f(A, B, \ldots, P)$, are all therefore obtained by assigning proper values to $i, j, \ldots, k$ in

$$f(a_i, b_j, \ldots, p_k). \quad (11)$$

We can therefore find n sets of constants $\alpha_i, \beta_i, \ldots, \rho_i$ (namely, the sets of values $a_i, b_j, \ldots, p_k$ in (11)) such that the roots of $f(A, B, \ldots, P)$ are

$$f(\alpha_i, \beta_i, \ldots, \rho_i), \quad i=1, 2, \ldots, n. \quad (12)$$

In particular, if $\lambda_1, \lambda_2, \ldots, \lambda_k$ are so chosen that $f(A, B, \ldots, P)=A$, we shall have $f(\alpha_i, \beta_i, \ldots, \rho_i)=\alpha_i$. The roots of A are therefore $\alpha_1, \alpha_2, \ldots, \alpha_n$. Similarly, the roots of B are $\beta_1, \beta_2, \ldots, \beta_n$, etc. By a change of notation we can then make $\alpha_i=a_i$, $\beta_i=b_i$, etc. The roots of $f(A, B, \ldots, P)$ are then $f(a_i, b_i, \ldots, p_i)$, which was to be proved.

5. *Canonical Form.*—Two sets of associated roots $a_1, b_1, \ldots, p_1$ and $a_2, b_2, \ldots, p_2$ will be considered different unless

$$a_1=a_2,\ b_1=b_2,\ \ldots,\ p_1=p_2.$$

Let $$a_i,\ b_i,\ \ldots,\ p_i, \quad i=1, 2, \ldots, m$$

be the different sets of roots of the commutative matrices $A, B, \ldots, P$. Let $\lambda, \mu, \ldots, \rho$ be numbers not satisfying any of the equations

$$\lambda a_i+\mu b_i+\ldots+\rho p_i=\lambda a_k+\mu b_k+\ldots+\rho p_k, \quad i\neq k.$$

By Theorem II, the distinct roots of $\lambda A+\mu B+\ldots+\rho P$ are

$$\lambda a_i+\mu b_i+\ldots+\rho p_i, \quad i=1, 2, \ldots, m.$$

The factors of its characteristic equation are

$$\lambda A+\mu B+\ldots+\rho P-(\lambda a_i+\mu b_i+\ldots+\rho p_i)I$$
$$=\lambda(A-a_iI)+\mu(B-b_iI)+\ldots+\rho(P-p_iI),$$

and the equation of least degree satisfied for all values of $\lambda, \mu, \ldots, \rho$ is

$$\prod_{i=1}^{m}[\lambda(A-a_iI)+\mu(B-b_iI)+\ldots+\rho(P-p_iI]^{r_i}=0, \tag{13}$$

the numbers $r_1, r_2, \ldots, r_m$ being the multiplicities of the roots.

Let $$\psi_1=\prod_{i=2}^{m}[\lambda(A-a_iI)+\mu(B-b_iI)+\ldots+\rho(P-p_iI)]^{r_i}$$

and, generally, let ψ_j be the product of all the factors in (13) except

$$[\lambda(A-a_jI)+\mu(B-b_jI)+\ldots+\rho(P-p_jI)]^{r_j}.$$

It is clear that

$$\psi_i\psi_k=0, \quad i\neq k, \tag{14}$$

for the product contains all the factors in the left member of (13). Let ψ_i' be the function obtained by replacing $\lambda, \mu, \ldots, \rho$ in ψ_i by $\lambda', \mu', \ldots, \rho'$. Consider the product

$$[\lambda'(A-a_1I)+\ldots+\rho'(P-p_1I)]^{r_1}\psi_1. \tag{15}$$

When this is multiplied by ψ_1' the result is zero because

$$[\lambda'(A-a_1I)+\ldots+\rho'(P-p_1I)]^{r_1}\psi_1'=0.$$

Also the product of (15) and any one of the functions $\psi_2, \psi_3, \ldots, \psi_n$ is zero by (14). Hence

$$[\lambda'(A-a_1I)+\ldots+\rho'(P-p_1I)]^{r_1}\psi_1(\psi_1'+\psi_2+\ldots+\psi_m)=0. \tag{16}$$

The roots of the matrix

$$\psi_1'+\psi_2+\ldots+\psi_m \tag{17}$$

are obtained by replacing $A, B, \ldots, P$ by $a_i, b_i, \ldots, p_i$. When they are replaced by $a_1, b_1, \ldots, p_1$ the result is not zero because ψ_1' is not zero, whereas

$$\psi_2=\psi_3=\ldots=\psi_m=0.$$

In a similar way it is seen that the other roots of (17) are not zero. We can therefore divide (16) by (17) and so get

$$[\lambda'(A-a_1I)+\ldots+\rho'(P-p_1I)]^{r_1}\psi_1=0. \tag{18}$$

Since $\lambda', \mu', \ldots, \rho'$ are arbitrary, if (18) is expanded in powers of those parameters, each coefficient will vanish. Hence

$$(A-a_1I)^{\alpha_1}(B-b_1I)^{\beta_1}\ldots(P-p_1I)^{\rho_1}\psi_1=0$$

for all positive integral values of $\alpha_1, \beta_1, \ldots, \rho_1$ such that

$$\alpha_1+\beta_1+\ldots+\rho_1=r_1.$$

In a similar way it is shown that

$$(A-a_iI)^{\alpha_i}(B-b_iI)^{\beta_i}\ldots(P-p_iI)^{\rho_i}\psi_i=0, \tag{19}$$

for all positive integral exponents such that

$$\alpha_i+\beta_i+\ldots+\rho_i=r_i. \tag{20}$$

Let
$$\psi=\psi_1+\psi_2+\ldots+\psi_m. \tag{21}$$

As in case of (17) it is shown that none of the roots of (21) are zero. We can then define a set of functions ϕ_i by the equations

$$\phi_i=\frac{\psi_i}{\psi}, \quad i=1, 2, \ldots, m. \tag{22}$$

It is clear that

$$\phi_1+\phi_2+\ldots+\phi_m=I. \tag{23}$$

Also, by (14),

$$\phi_i\phi_k=0, \quad i\neq k. \tag{24}$$

Multiplying (23) by ϕ_i we therefore get

$$\phi_i^2=\phi_i, \quad i=1, 2, \ldots, m. \tag{25}$$

Let $f(a, b, \ldots, p)$ be a polynomial in $a, b, \ldots, p$ and

$$f_{\alpha,\beta,\ldots,\rho}(a, b, \ldots, p)=\frac{\partial^{\alpha+\beta+\ldots+\rho}}{\partial a^\alpha\partial b^\beta\ldots\partial p^\rho}f(a, b, \ldots, p).$$

By Taylor's theorem

$$f(a, b, \ldots, p)=f(a_i, b_i, \ldots, p_i)$$
$$+\sum_\alpha\sum_\beta\ldots\sum_\rho f_{\alpha,\beta,\ldots,\rho}(a_i, b_i, \ldots, p_i)\frac{(a-a_i)^\alpha}{\underline{|\alpha}}\ldots\frac{(p-p_i)^\rho}{\underline{|\rho}}.$$

Since this is an identity in the variables $a, b, \ldots, p$, we can replace them by $A, B, \ldots, P$. Then

$$f(A, \ldots, P) = f(a_i, \ldots, p_i)I + \sum_{\alpha} \ldots \sum_{\rho} f_{\alpha, \ldots, \rho}(a_i, \ldots, p_i) \frac{(A-a_iI)^\alpha}{\lfloor\alpha} \ldots \frac{(P-p_iI)^\rho}{\lfloor\rho}. \quad (26)$$

Let $\qquad A_i = \phi_i(A-a_iI), \ldots, P_i = \phi_i(P-p_iI).$

From (25) it follows that

$$A_i^\alpha B_i^\beta \ldots P_i^\rho = \phi_i(A-a_iI)^\alpha(B-b_iI)^\beta \ldots (P-p_iI)^\rho.$$

Equations (19), (20), and (22) show that this is zero if $\alpha+\beta+\ldots+\rho \geqq r_i$. Finally, if we multiply (26) by ϕ_i, sum for $i=1, 2, \ldots, m$, and use (23) we get

$$f(A, \ldots, P) = \sum_{i=1}^{m} \phi_i f(a_i, \ldots, p_i) + \sum_{i=1}^{m} \sum_{\alpha} \ldots \sum_{\rho} f_{\alpha, \ldots, \rho}(a_i \ldots p_i) \frac{A_i^\alpha \ldots P_i^\rho}{\lfloor\alpha \ldots \lfloor\rho}, \quad (27)$$

the summation including powers $A_i^\alpha \ldots P_i^\rho$ for which

$$\alpha+\beta+\ldots+\rho < r_i. \quad (28)$$

Equation (27) gives a form in which any polynomial in the given commutative matrices $A, B, \ldots, P$ can be expressed. We shall refer to it as the canonical form for a function of the matrices.* Its most important property is expressed in the following theorem:

THEOREM III. *If $A, B, \ldots, P$ are commutative matrices with corresponding roots $a_i, b_i, \ldots, p_i$ and $f(a, b, \ldots, p)$ is any polynomial, the matrix $f(A, B, \ldots, P)$ can be expressed as a linear function of matrices depending only on $A, B, \ldots, P$, the coefficients in the linear functions being obtained by substituting each set of roots $a_i, b_i, \ldots, p_i$ in $f(a, b, \ldots, p)$ and in its partial derivatives of order lower than the multiplicity of the root $\lambda a_i + \mu b_i + \ldots + \rho p_i$ in the equation of least degree satisfied for all values of $\lambda, \mu, \ldots, \rho$ by $\lambda A + \mu B + \ldots + \rho P$.*

A case of particular interest is that of a polynomial such that all the coefficients $f(a_i, \ldots, p_i)$, $f_{\alpha, \ldots, \rho}(a_i, \ldots, p_i)$ in (27) vanish. Then, evidently, $f(A, B, \ldots, P) = 0$. We can consider the values $a_i, b_i, \ldots, p_i$ as coordinates of a point in hyperspace. In the case considered, the function $f(a, b, \ldots, p)$ has a zero of order r_i at the point $(a_i, b_i, \ldots, p_i)$. Therefore we have proved

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* The formula for a function of a single matrix with distinct roots was given by Sylvester, *Comptes Rendus*, Vol. XCIV (1882), p. 55. The case of a single matrix with repeated roots was given by A. Buchheim, *Phil. Mag.*, (5) 22 (1886), pp. 173–174.

THEOREM IV. *If the polynomial $f(a, b, \ldots, p)$ has at each of the points $(a_i, b_i, \ldots, p_i)$, $i=1, 2, \ldots, m$, a zero of order equal to or greater than the multiplicity of the corresponding root in the equation of least degree satisfied for all values of $\lambda, \mu, \ldots, \rho$ by $\lambda A+\mu B+\ldots+\rho P$ then $f(A, B, \ldots, P)=0$.*

If the matrices ϕ_i, $A_i^\alpha B_i^\beta \ldots P_i^\rho$ in (27) are linearly independent, Theorem IV expresses the necessary and sufficient condition that $f(A, B, \ldots, P)$ vanish. This is easily shown to be true for polynomials in a single matrix.* For polynomials in two or more matrices such may not be the case. For instance, A and B could be equal. Then A_i and B_i would be equal.

6. *Commutative Matrices not Expressible as Polynomials in the same Matrix.*—The simplest illustration of commutative matrices is furnished by polynomials in a single matrix. If

$$A=\alpha_1\phi^p+\alpha_2\phi^{p-1}+\ldots+\alpha_p I, \quad B=\beta_1\phi^q+\beta_2\phi^{q-1}+\ldots+\beta_i I,$$

obviously A and B are commutative. If it were true that any two commutative matrices could be so expressed,† by a repetition of the process, any finite number could be expressed as polynomials in the same matrix. The results of the preceding sections could then be more readily obtained by using these expressions. That such is not the case will now be shown by a simple example. Let

$$A=\begin{Vmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix}, \quad B=\begin{Vmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix}.$$

By direct multiplication it is shown that

$$A^2=B^2=AB=BA=0. \tag{29}$$

Hence A and B are commutative. Suppose they are expressible as polynomials in ϕ. Since the characteristic equation for ϕ is of third degree, the expressions can be reduced to the form

$$A=\alpha_1\phi^2+\alpha_2\phi+\alpha_3 I, \quad B=\beta_1\phi^2+\beta_2\phi+\beta_3 I. \tag{30}$$

If α_1 and β_1 are both zero, these polynomials are of first degree. If not, we form the expression

$$(\beta_1 A-\alpha_1 B)^2=[\beta_1(\alpha_2\phi+\alpha_3 I)-\alpha_1(\beta_2\phi+\beta_3 I)]^2. \tag{31}$$

Equation (29) shows that this is zero. The expression

$$[\beta_1(\alpha_2 x+\alpha_3)-\alpha_1(\beta_2 x+\beta_3)]^2$$

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* See Metzler, AMERICAN JOURNAL, Vol. XIV (1892), p. 339.

† Frobenius raised this question in the article to which we have previously referred, but stated that he had not decided whether it could be done or not.

is not identically zero, however, because A and B do not satisfy an equation of the form $\beta_1 A - \alpha_1 B = 0$. By the use of (31) we can then reduce (30) to the form

$$A = a_1\phi + a_2 I, \quad B = b_1\phi + b_2 I. \tag{32}$$

The coefficients a_1 and b_1 can not be zero because A and B are not multiples of the unit matrix. Hence, as before, we form the expression

$$(b_1 A - a_1 B)^2 = (b_1 a_2 - a_1 b_2)^2 I = 0.$$

This shows that $b_1 a_2 - a_1 b_2 = 0$, and so, from (32), $b_1 A - a_1 B = 0$. Since A is not a multiple of B this is impossible. Therefore A and B cannot be expressed as polynomials in ϕ.

7. *Limits and Convergence.*—A variable matrix $\phi = \|r_{ik}\|$ is said to approach a matrix $A = \|a_{ik}\|$ as limit if

$$\operatorname{Lim} r_{ik} = a_{ik}, \qquad i, k = 1, 2, \ldots, n.$$

In this case it is clear that each root of ϕ will approach a root of A as limit. For the roots of ϕ and A satisfy the equations

$$|rI - \phi| = r^n + \rho_1 r^{n-1} + \ldots + \rho_n = 0, \tag{33}$$

$$|aI - A| = a^n + \alpha_1 a^{n-1} + \ldots + \alpha_n = 0. \tag{34}$$

The coefficients $\rho_1, \rho_2, \ldots, \rho_n$ are definite rational integral functions of the elements r_{ik} of ϕ, and the coefficients $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the same functions of the elements of A. When ϕ approaches A as limit

$$\operatorname{Lim} \rho_i = \alpha_i, \qquad i = 1, 2, \ldots, n.$$

Also, if the coefficient of the highest power (in this case unity) does not approach zero, the roots of a polynomial equation are continuous functions of its coefficients.* Hence as ϕ approaches A as limit the roots of (33) approach those of (34) as limits.

Since different matrices can have the same roots, the roots of ϕ can approach those of A when ϕ does not approach A as limit.

The sum $\phi_1 + \phi_2 + \ldots + \phi_m$ of m matrices is equal to a single matrix S_m If S_m approaches a limit S when m increases indefinitely, the series

$$\phi_1 + \phi_2 + \ldots + \phi_m + \ldots$$

is said to converge and have the sum S.† In case of a multiple series, such as

$$\Sigma \phi_{i,j,k}, \qquad i, j, k = 1, 2, \ldots, \infty,$$

* Weber's Algebra, Vol. I, § 44.

† Peano treated the convergence of a matrix series by means of a modulus, *Math. Annalen* (1888), Vol. XXXII, pp. 450–456. See also G. A. Bliss, ***Annals of Mathematics*** (2) 6, pp. 49–68.

an order of summation is assumed to be included in the definition of the series. It is thus equivalent to a simple series.

If $A, B, \ldots, P$ are commutative matrices, by Theorem III, any polynomial $f(A, B, \ldots, P)$ is expressible as a linear function of matrices depending only on $A, B, \ldots, P$, the coefficients being $f(a_i, b_i, \ldots, p_i)$ and perhaps some of the derivatives $f\alpha, \beta, \ldots, \rho(a_i, b_i, \ldots, p_i)$. If these coefficients approach definite limits when the number of terms in the polynomial increases indefinitely, the matrix $f(A, B, \ldots, P)$ will then approach a definite limit.

THEOREM V. *If $A, B, \ldots, P$ are commutative matrices and $f(a, b, \ldots, p)$ an infinite series, the series $f(A, B, \ldots, P)$ will converge and be represented by* (27) *if the series $f(a_i, b_i, \ldots, p_i)$, $f_{\alpha, \beta, \ldots, \rho}(a_i, b_i, \ldots, p_i)$ in the right member of* (27) *converge.*

In this theorem it is understood that $f_{\alpha, \beta, \ldots, \rho}(a, b, \ldots, p)$ is obtained by differentiating $f(a, b, \ldots, p)$ term by term and arranging the results in the same order as the corresponding terms in $f(a, b, \ldots, p)$.

If $f(a, b, \ldots, p)$ is an analytic function of the variables $a, b, \ldots, p$ and $A, B, \ldots, P$ an equal number of commutative matrices, we define $f(A, B, \ldots, P)$ as the value (if it is definite) determined by (27). In order that the value be definite it is sufficient that $f(a, b, \ldots, p)$ be analytic near each of the points $(a_i, b_i, \ldots, p_i)$ determined by the distinct sets of associated roots. For then the derivatives in (27) are also definite. If the function is many valued, a definite branch must be chosen at each of the points $(a_i, b_i, \ldots, p_i)$. The same branch need not however be used for all.

8. *Taylor's Series.*—Let $f(z)$ be a function of the complex variable z analytic in the neighborhood of each root of a matrix A. If we choose a definite branch of $f(z)$ at each root (but not necessarily the same branch for different roots) the functions $f(A)$, $f'(A)$, $f''(A)$, etc. are determined by (27). Let Z be a matrix commutative with A and consider the series

$$\psi(A, Z) = f(A) + f'(A)(Z-A) + \ldots + f^m(A)\frac{(Z-A)^m}{\underline{|m}} + \ldots \quad (35)$$

Let z_i and a_i be corresponding roots of Z and A. The series (35) will converge if each root z_i lies within the circle of convergence of $f(z)$ with center at a_i. For then the series

$$f(a_i) + f'(a_i)(z_i - a_i) + \ldots + f^m(a_i)\frac{(z_i - a_i)^m}{\underline{|m}} + \ldots, \quad (36)$$

and the series obtained by substituting a_i and z_i in the partial derivatives of $\psi(x, y)$ will converge. Hence the conditions of Theorem V are satisfied.

Conversely, if (35) converges, (36) will converge, and so z_i will lie within or on the circle of convergence of $f(z)$ with center a_i. For the sum of m terms in (36) represents a root of the sum of corresponding terms in (35), and, if a matrix approaches a limit, we have shown that its roots must approach definite limits. Furthermore

$$\psi(a_i, z_i) = f(a_i) + f'(a_i)(z_i - a_i) + \ldots + f^m(a_i)\frac{(z_i - a_i)^m}{\underline{|m}} + \ldots = f(z_i).$$

Also, if

$$\psi_{\alpha,\beta}(x, y) = \frac{\partial^{\alpha+\beta}\psi(x, y)}{\partial x^\alpha \partial y^\beta}$$

it is readily seen that

$$\psi_{\alpha,0}(a_i, z_i) = f^\alpha(z_i), \quad \psi_{\alpha,\beta}(a_i, z_i) = 0, \quad \beta \neq 0.$$

Hence (27) gives the same expression for $\psi(A, Z)$ and for $f(Z)$. Therefore

$$f(Z) = f(A) + f'(A)(Z - A) + \ldots + f^m(A)\frac{(Z - A)^m}{\underline{|m}} + \ldots. \tag{37}$$

We may call this the expansion of $f(Z)$ in the neighborhood of the matrix A.

THEOREM VI. *The Taylor's series* (37) *for* $f(Z)$ *is valid for any matrix* Z *commutative with* A *if each root of* Z *lies within a circle with center at the corresponding root of* A, *in which* $f(z)$ *is analytic.*

This theorem enables us to determine an analytic function of a matrix Z by means of a power series

$$c_1 + c_2(Z - A) + \ldots + c_m(Z - A)^m + \ldots,$$

and its analytic continuations just as we determine an analytic function of the complex variable z by the series

$$c_1 + c_2(z - a) + \ldots + c_m(z - a)^m + \ldots,$$

and its analytic continuations. The principal difference in the two cases is that Z and A must be commutative and that the region of convergence of the matrix series consists of m circles (one for each distinct root of A) instead of one. On a range of commutative matrices the function $f(Z)$ is one valued if each root of Z is restricted to a region of the complex plane (not necessarily the same for different roots) in which the corresponding branch of $f(z)$ is one valued.

On the Lüroth Quartic Curve.

By Frank Morley.

It has been known since 1870 * that the problem of inscribing a five-line in a planar quartic is poristic; of the ten conditions nine fall on the lines and one on the curve. Thus the quartic is one for which an invariant vanishes, and the degree of this invariant is sought. We use Aronhold's construction of a curve of class 4 from seven given points. And the starting point is the theorem of Prof. Bateman † that the seven points which have the same polar line as to a conic and a cubic give rise to a Lüroth quartic.

For completeness I indicate the proof. A conic and a cubic have the canonical forms (αx^2), (βx^3) where $(x)=0$. The polars of x are (αxy), (βxy^2). Working in a space of three dimensions the line $(y)=0$, $(\alpha xy)=0$ is to touch the quadric (βxy^2). This requires that

$$\Sigma\beta_0\beta_1 x_0 x_1(\alpha_2 x_2-\alpha_3 x_3)^2=0,$$

or

$$(\alpha/\beta)^2/(\alpha^2 x/\beta)=(1/\beta x),$$

and this is a quartic of Lüroth's type. The seven common polar lines are an Aronhold set of double lines of this quartic, and by polarity as to the conic the seven points a_i which have these polar lines are double points of a Lüroth curve of class 4.

§ 1. *The Bateman Conic.*

Take now a conic $(\alpha x)^2$ and a cubic $(\beta x)^3$. The Jacobian of these and a line (ξx)

$$(\alpha x)(\beta x)^2|\alpha\beta\xi|=0$$

gives the net of cubics on the seven points a_i. Referred to one of the points and the corresponding line let the conic be $x_0^2+2x_1x_2$ and the cubic be

$$x_0^3+x_0(\gamma x)_0^2+(\delta x)_0^3.$$

Then for $(\xi x)\equiv x_0$ the Jacobian is

$$(\alpha_1\beta_2-\alpha_2\beta_1)(\alpha x)(\beta x)^2=\beta_2(\beta x)^2x_2-\beta_1(\beta x)^2x_1,$$

so that not only terms in x_0^2 but also the term $x_0x_1x_2$ is missing.

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* Lüroth, *Math. Annalen*, Vol. I.

† American Journal of Mathematics, Vol. XXXVI.

That is, *the seven cubics with double points a_i have their nodal tangents apolar to the conic α.* I will call this conic the Bateman conic.

Given any seven points a_i, cubics on them determine by their remaining intersections a Geiser involution $a^8x^8\xi=0$. If ξ is the join of x and y, then $a^8x^9y=0$, or removing the Jacobian of the net, an a^3x^6, $a^5x^3y=0$. This is the canonic form of the net.*

It may be written as a two-one connex $a^5x^2\xi$, giving for every line ξ the Geiser pair on ξ. This Geiser pair is the neutral pair of the net of binary cubics of ξ, cut out by the net of cubic curves. The quartic, locus of lines ξ for which the Geiser pair come together, is found by making ξ touch the conic $a^5x^2\xi$, and is an $a^{10}\xi^4$.

Write the above two-one connex $a^5x^2\xi$ as $(\gamma x)^2(c\xi)$, and consider $(\gamma x)(\gamma y)\,|cxy|$ an $a^5x^2y^2$. This skew form is the polar conic of y as to its associate cubic, and when y is a_i it is the nodal tangents to the cubic with double point a_i. We have seen that in the case in question these seven line pairs are apolar to a conic. But there are only six independent conics. Thus the required condition is that for arbitrary y the associate conic be apolar to a conic. That is, the six-rowed determinant of all coefficients $\gamma_{ij}c_k$ vanishes. But being a skew determinant it is a square. Thus *a cubic function of the coefficients $\gamma_{ij}c_k$ vanishes.*

§ 2. *The Cubic Invariant of Seven Points.*

For the connex $(\gamma x)^2(c\xi)$ the possible expressions of the third degree are

$$(c\gamma)(c'\gamma')(c''\gamma'')\,|\gamma\gamma'\gamma''|,\quad (c\gamma)(c'\gamma'')(c''\gamma')\,|\gamma\gamma'\gamma''|,$$
$$(c\gamma')(c'\gamma'')(c''\gamma)\,|\gamma\gamma'\gamma''|,\quad |cc'c''|\,|\gamma\gamma'\gamma''|^2.$$

Of these the first, second, and fourth change sign on interchange of $c'\gamma'$ with $c''\gamma''$, and therefore are zero. Thus the invariant in question is

$$(c\gamma')(c'\gamma'')(c''\gamma)\,|\gamma\gamma'\gamma''|.$$

The invariant expressed in terms of the seven points a_i is of degree 15. But if six points are on a conic it will vanish.

For the form $a^5x^2y^2$ was made up in this way: on the line $\overline{xy}$ is a net of binary cubics, with a neutral pair, and xy are taken harmonic with this neutral pair. If $\overline{ya_i}$ meet the cubic curve with double point at a_i at p_i, then the

*An expression for the net of cubics on seven given points may be noted, though not of present use. Let A be the Jacobian of cubics on $a_2 \dots\dots a_7x$, and so on. Then the determinant of seven rows

$$|\,a^2_{i0},\ a^2_{i1},\ a^2_{i2},\ a_{i1}a_{i2},\ a_{i2}a_{i0},\ a_{i0}a_{i1},\ (yDa_i)A_0\,|$$

is the expression in question. For when $x=a_1$, $A_2 \dots\dots A_7$ vanish and $(yDa_1)A_1$ also vanishes. Presumably this expression for the net is canonic.

neutral pair is a_i and p_i, and if the polar of y as to these be x_i, then the seven points x_i are on the conic $a^5x^2y^2$ associate with y.

If now $a_2 \ldots . a_7$ are on a conic $(\alpha x)^2$, the points $p_2 \ldots . p_7$ are on the line $(\alpha x)(\alpha y)$, and the form $a^5x^2y^2$ becomes $(\alpha x)(\alpha y)\,|a_1xy|$.

This with a_1 as the reference point $(1, 0, 0)$ and $(\alpha x)^2$ as $x_0^2+2x_1x_2$ becomes

$$(x_0y_0+x_1y_2+x_2y_1)\,(x_1y_2-x_2y_1),$$

and since there is no term in y_0^2 the invariant of the coefficients vanishes. Thus the expression of degree 15 in a_i breaks up, and removing the factors which vanish when any six of the seven points are on a conic, we are left with *a cubic expression in the* a_i.

Thus, given six of the points, the locus of the seventh, a_7, is a cubic curve. If once more the six are on a conic $(\alpha x)^2$ then the nodal tangents of the cubic $|a_ia_7x|\,(\alpha x)^2$ are apolar to $(\alpha x)^2$, which is therefore the Bateman conic. Then the tangents at a_7 to the cubic of the system with double point a_7 are apolar to the conic, and this defines the cubic. Thus, if the six points on the conic be given by the binary form $(\beta t)^6$, the locus of a_7 is $(\beta t)^3(\beta t')^3=0$, namely, that cubic on the six points to which the conic (as a line-curve) is apolar.

Thus a special seven-point for which the cubic invariant vanishes is six points on a conic and any point on the apolar cubic through them.

Hence, given any six points $a_1 \ldots . a_6$ we have a counter-six $b_1 \ldots . b_6$ where b_1 is the extra point in which the conic on $a_2 \ldots . a_6$ meets the cubic on $a_1 \ldots . a_6$ and apolar to this conic. The locus of a_7 passes through all twelve points. It is to be noticed that the relations of the points a_i and the points b_i are mutual.

Expressed in terms of Professor Coble's* linear invariants $\bar{a} \ldots . \bar{f}$ of six points, and linear covariants $a \ldots . f$, these being cubics on the points, the covariant cubic in question can be no other than $\Sigma\bar{a}^2a$. This is then an expression for the cubic invariant of seven points.

§ 3. *The Lüroth Invariant.*

If now we map the plane on a cubic surface by means of cubics on the six points $a_1 \ldots . a_6$, the covariant cubic curve becomes a covariant plane of the isolated double-six of lines on the surface. The construction becomes as follows: Let a_i and b_i be a pair of lines of the double-six. Sections of the surface on a_i determine points on b_i. The tangent conic sections determine two points on b_i. There is a conic on the surface through these two points and this determines a point on a_i, on the plane required.

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* *Transactions*, Vol. XVI (1915), § 4.

If from any point where this plane meets the surface we draw the tangent lines we obtain a quartic curve of Lüroth's type. Now a cubic surface has thirty-six double-sixes, and therefore thirty-six such planes. The locus of points on the surface which give rise to Lüroth quartics is then thirty-six planar cubics.

But a covariant of the surface of order 2μ gives an invariant of the corresponding quartic of degree 3μ. Hence, *the Lüroth invariant is of degree* 54.

A line of the surface belongs to sixteen double sixes. Thus the thirty-six planes meet a line of the surface in $16+20$ points. Thus a Lüroth quartic can acquire a double point in two ways. In the one, the lines at the double point are apolar to the points on a double line. In the other the lines at the double point meet the curve again on a line of the curve.

This indicates the nature of the Lüroth invariant I_{54}, namely it is, to the discriminant I_{27} as modulus, the product of two invariants.

Consider a nodal cubic surface in Sylvester's form, $(\varkappa x^3)$ where $(1/\sqrt{\varkappa})=0$. It can be proved that the plane corresponding to the double-six of lines on the node is $(x\sqrt{\varkappa})$.

Hence, when $(1/\sqrt{\varkappa})$ is not 0, there is a covariant of order 16, product of the sixteen planes $(x\sqrt{\varkappa})$, meeting any line of the surface at the sixteen points on it, so that there is an invariant I_{24} which vanishes for a nodal Lüroth quartic of the first kind.

For a nodal Lüroth quartic of the second kind, the inscribed five lines are three lines on the double point, and two lines on a fixed point of the curve. In particular the tangents from the node fall into two sets of three, each set having its contacts on a line. Thus such a quartic is included in those for which three intersections of double lines lie on a line.

Looking then at a double-six from a point y of its cubic surface, the six lines from y to each pair lie on a quadric cone which breaks up into two planes when y is on one of ten planes corresponding to the separation of the six pairs into threes.

If in the case of a nodal cubic $(\varkappa x^3)$ surface these ten planes formed for the nodal double-six have an equation rational in $\varkappa_i$, then an invariant I_{15} vanishes for the nodal Lüroth quartic of the second kind, and the Lüroth invariant will be $I_{27}I_{27}^1+I_{24}I_{15}^2$ where I_{27}^1 is an invariant which is probably the discriminant also.

VINEYARD HAVEN, MASS., *July*, 1918.

*On the Order of a Restricted System of Equations.**

BY F. F. DECKER.

Section 1.

In an earlier paper † the writer gave a proof of the theorem—announced without proof by Salmon ‡—that the number of solutions of the system of equations arising from the vanishing of all the determinants of the m-th order that can be formed from a matrix with m rows and n columns, $m \ngtr n$, by the suppression of $n-m$ columns, the elements in the i-th row and j-th column being of degree α_i+a_j in $n-m+1$ non-homogeneous variables is K_{n-m+1} where $K_l=\sum_{i=0}^{i=l} c_i\delta_{l-i}$, c_l representing the sum of all possible products of l different a's and δ_l the sum of all possible products of l α's, repetitions being permissible.

In this paper the system arising from the vanishing of all the determinants of the r-th order that can be formed by the suppression of $n-r$ columns and $m-r$ rows, $r \ngtr m$, $r \ngtr n$, is treated. The order is shown to be

$$\begin{vmatrix} K_{n-r+1} & K_{n-r} & \ldots K_{n-m+1} \\ K_{n-r+2} & K_{n-r+1} & \ldots K_{n-m+2} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ K_{n-2r+m+1} & K_{n-2r+m} & \ldots K_{n-r+1} \end{vmatrix} \tag{A}$$

If the degree of every element is b the order reduces to

$$\frac{\prod_{i=0}^{i=m-r} {}_{n+m-r}C_{m+1-i}}{\prod_{i=0}^{i=m-r} {}_{n+m-r}C_i} \cdot b^{(n-r+1)(m-r+1)}, \text{ which may be written}$$

$$\frac{\prod_{i=0}^{i=m-r} {}_{n+i}C_{n-r+1}}{\prod_{i=0}^{i=m-r} {}_{n-r+1+i}C_{n-r+1}} \cdot b^{(n-r+1)(m-r+1)}, \tag{B}$$

a result established by Segre. §

* Presented before the American Mathematical Society, April 27, 1918.

† "On the Order of a Restricted System of Equations," AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXVII, No. 2, April, 1915.

‡ Salmon, "Modern Algebra," Fourth Edition, pp. 283–313.

§ Segre, "Gli ordini delle varietà che annullano dei diversi gradi estratti da una data matrice," *Rendic. R. Accad. Dei Lincei*, Series 5, Vol. IX, session of October 21, 1900.

In § 2 the notation is defined and some preliminary relations are given. Theorem VI simplifies the derivation of the order for $m > n$ after it has been derived for $m \not> n$ (Theorem XII). In § 3 is considered the case of the vanishing of the determinants of two matrices having some common rows. The order of that part of the system for which the determinants of the common rows vanish is found (Theorem IX). In § 4 is established by mathematical induction, with the aid to Theorem V, the relation (A) of § 1 (Theorem XIII), and the relation (B) is established in § 5 (Theorem XIV).

Section 2.

$\left\|\begin{matrix} u_{11} \dots u_{1n} \\ \dots\dots\dots \\ u_{m1} \dots u_{mn} \end{matrix}\right\|^{(r)}$ or $\|U_{mn}\|^{(r)}$ or $\begin{pmatrix} 1 \dots n \\ 1 \dots m \end{pmatrix}^{(r)}$ denotes the aggregate of all determinants of the r-th order that can be formed from the matrix $\left\|\begin{matrix} u_{11} \dots u_{1n} \\ \dots\dots\dots \\ u_{m1} \dots u_{mn} \end{matrix}\right\|$ by suppressing $m-r$ rows and $n-r$ columns. That all these determinants vanish will be indicated by

$$\left\|\begin{matrix} u_{11} \dots u_{1n} \\ \dots\dots\dots \\ u_{m1} \dots u_{mn} \end{matrix}\right\|^{(r)} = 0 \text{ or } \|U_{mn}\|^{(r)} = 0 \text{ or } \begin{pmatrix} 1 \dots n \\ 1 \dots m \end{pmatrix}^{(r)} = 0.$$

$M(\bar{h}, \bar{k})$ denotes the matrix formed from $\begin{pmatrix} 1 \dots n \\ 1 \dots m \end{pmatrix}$ by suppressing the first h rows and the last k rows, that is $\begin{pmatrix} 1 \dots n \\ h+1 \dots m-k \end{pmatrix}$.

The order of the system of equations $A=0$ will be indicated by $\bar{A}$. When the (r) is lacking from the symbol it will be understood to be the same as the smaller of the two numbers m, n. If r is one more than the smaller of the numbers m, n, the order symbol is to be understood to represent the number 1, and if r exceeds each of the numbers m, n by more than one, the order symbol is to be understood to represent zero.

ϕ is an operator such the $\phi \bar{M}(\bar{h}_1, \bar{k}_1) \bar{M}(\bar{h}_2, \bar{k}_2) \dots \bar{M}(\bar{h}_s, \bar{k}_s)$ produces the algebraic sum of all the products that can be formed from

$$\bar{M}(\bar{h}_1, \bar{k}_1) \bar{M}(\bar{h}_2, \bar{k}_2) \dots \bar{M}(\bar{h}_s, \bar{k}_s)$$

by interchanging k_1 with each of the other k's in turn, the sign of each product being given by the formula $(-1)^{t_i+1}$ where t_i is the number of inversions of

the natural order of the k's in the different permutations, the h's occurring in the natural order. $K_l \equiv \sum_{i=0}^{i=l} c_i\delta_{l-i}$ and $J_l \equiv \sum_{i=0}^{i=l} d_i\gamma_{l-i}$ where $c_l \equiv \sum_1^s \prod_{i=1}^{i=l} \alpha_i$, $\gamma_l \equiv \sum_1^s \prod_{i=1}^{i=l} \alpha_i$, and d and δ differ from c and γ, respectively, only in permitting repetitions of the same letter in forming homogeneous products. $K_0=J_0=c_0=d_0=\gamma_0=\delta_0=1$. When l is negative the values of the symbols are zero. $K'_s=K_{n-m+s}$.

u_{rs} denotes a function of not less than $(n-r+1)(m-r+1)$ non-homogeneous variables, and, whenever the order is calculated in terms of K's or J's, u_{rs} is considered to be of degree $\alpha_r+\alpha_s$.

Use will be made of the following relations (Theorems I–III), proofs of which may be found in a previous paper on the subject by the writer already referred to.

THEOREM I. $\overline{\begin{pmatrix} 1\ldots.n \\ 1\ldots.m \end{pmatrix}} = K_{n-m+1}$.

THEOREM II. $\overline{\begin{pmatrix} 1\ldots.n \\ 1\ldots.m \end{pmatrix}} = \sum_1^m (-1)^{i+1} \overline{\begin{pmatrix} i+1\ldots.n \\ 1\ldots.m \end{pmatrix}}^{(m)} \overline{\begin{pmatrix} i\ldots.m \\ 1\ldots.m \end{pmatrix}}^{(m-i+1)}$

$$= \sum_1^m (-1)^{i+1} \overline{\begin{pmatrix} 1\ldots.n-i \\ 1\ldots.m \end{pmatrix}}^{(m)} \overline{\begin{pmatrix} n-m+1\ldots.n-i+1 \\ 1\ldots.m \end{pmatrix}}^{(m-i+1)}.$$

THEOREM III. $\sum_{i=0}^{i=l} (-1)^i K_{l-i} J_i = 0$.

THEOREM IV. $\underset{a=r+1\ldots m}{K_n} = K_n - K_{n-1}\sum_1^r \alpha_1 + K_{n-2}\sum_1^r \alpha_1\alpha_2 - \ldots + (-1)^r K_{n-r} \sum_1^r \alpha_1\alpha_2\ldots.\alpha_r$, *where the a's run from* 1 *to* n *in all the* K's, *and the* α's *run from* 1 *to* m, *except in the case noted.*

To establish this relation we divide the terms of K_n into those that contain α_1 and those that do not. From the definition of K, $K_n = \alpha_1 K_{n-1} + \underset{a=2\ldots.m}{K_n}$, whence $\underset{a=2\ldots.m}{K_n} = K_n - \alpha_1 K_{n-1}$. Then we assume that

$$\underset{r\ldots.m}{K_n} = K_n - K_{n-1}\sum_1^{r-1}\alpha_1 + \ldots. + (-1)^{l-1}K_{n-l+1}\sum_1^{r-1}\alpha_1\alpha_2\ldots.\alpha_{l-1}$$
$$+ (-1)^l K_{n-l}\sum_1^{r-1}\alpha_1\alpha_2\ldots.\alpha_l + \ldots.$$

It follows that

$$\underset{r+1\ldots.m}{K_n} = \underset{r\ldots.m}{K_n} - \alpha_r \underset{r\ldots.m}{K_{n-1}} = K_n - (\alpha_r + \sum_1^{r-1}\alpha_1)K_{n-1}$$
$$+ \ldots. + (-1)^s(\alpha_r\sum_1^{r-1}\alpha_1\alpha_2\ldots.\alpha_{s-1} + \sum_1^{r-1}\alpha_1\alpha_2\ldots.\alpha_s)K_{n-s} + \ldots.$$
$$= K_n - K_{n-1}\sum_1^r\alpha_1 + \ldots. + (-1)^s K_{n-s}\sum_1^r\alpha_1\alpha_2\ldots.\alpha_s$$
$$+ \ldots. + (-1)^r K_{n-r}\sum_1^r\alpha_1\alpha_2\ldots.\alpha_r.$$

THEOREM V. $(1-\phi)\begin{pmatrix}\overline{1\ldots n}\\ s+1\ldots m\end{pmatrix}\Delta_s=\Delta_{s+1},\ m \not> n,$

$$\text{where } \Delta_l=\begin{vmatrix}\begin{pmatrix}\overline{1\ldots n}\\ 1\ldots m-l\end{pmatrix} & \begin{pmatrix}\overline{1\ldots n}\\ 1\ldots m-l+1\end{pmatrix} & \cdots & \begin{pmatrix}\overline{1\ldots n}\\ 1\ldots m-1\end{pmatrix}\\ \begin{pmatrix}\overline{1\ldots n}\\ 2\ldots m-l\end{pmatrix} & \begin{pmatrix}\overline{1\ldots n}\\ 2\ldots m-l+1\end{pmatrix} & \cdots & \begin{pmatrix}\overline{1\ldots n}\\ 2\ldots m-1\end{pmatrix}\\ \cdots & \cdots & \cdots & \cdots\\ \begin{pmatrix}\overline{1\ldots n}\\ l\ldots m-l\end{pmatrix} & \begin{pmatrix}\overline{1\ldots n}\\ l\ldots m-l+1\end{pmatrix} & \cdots & \begin{pmatrix}\overline{1\ldots n}\\ l\ldots m-1\end{pmatrix}\end{vmatrix}$$

$$=\begin{vmatrix}\bar M(\bar 0,\bar l) & \bar M(\bar 0,\overline{l-1}) & \ldots\bar M(\bar 0,\bar 1)\\ \bar M(\bar 1,\bar l) & \bar M(\bar 1,\overline{l-1}) & \ldots\bar M(\bar 1,\bar 1)\\ \cdots & \cdots & \cdots\\ \bar M(\overline{l-1},\bar l) & \bar M(\overline{l-1},\overline{l-1}) & \ldots\bar M(\overline{l-1},\bar 1)\end{vmatrix}.$$

Proof: $\Delta_s=\Sigma(-1)^{t_i+1}\bar M(0,k_1)\bar M(\bar 1,\bar k_2)\ldots\bar M(\overline{s-1},\bar k_s)$, where k_1, $k_2,\ldots,k_s$ is a permutation of the numbers $1, 2, \ldots, s$, and t_i the number of inversions of the natural order of the k's in the different permutations. When ϕ operates on $\bar M(\bar s,\bar 0)\Delta_s$ and the series of first indices is restored to the natural order, all the permutations of the second indices except $k_1, k_2, \ldots, k_s, 0$ will be obtained and the number of inversions of the series of second indices will be changed by one. Therefore

$$(1-\phi)\bar M(\bar s,\bar 0)\Delta_s=\Sigma(-1)^{t_i+1}\bar M(\bar 0,\bar k_1)\bar M(\bar 1,\bar k_2)\ldots\bar M(\bar s,\bar k_{s+1})=\Delta_{s+1}.$$

THEOREM VI. $\Delta=\Delta'$,*

$$\text{where } \Delta\equiv\begin{vmatrix}K_s & K_{s-1} & \ldots K_1\ 1\ \ 0\ 0\ldots 0 & 0\\ K_{s+1} & K_s & \ldots K_2\ K_1\ 1\ 0\ldots 0 & 0\\ \cdots & \cdots & \cdots & \cdot\\ K_{t-1} & K_{t-2} & \ldots\ldots K_1 & 1\\ K_t & K_{t-1} & \ldots\ldots K_2 & K_1\\ \cdots & \cdots & \cdots & \cdots\\ K_{s+t-1} & K_{s+t-2} & \ldots\ldots K_{s+1} & K_s\end{vmatrix} \text{ and } \Delta'\equiv\begin{vmatrix}J_t & J_{t-1} & \ldots J_{t-s+1}\\ J_{t+1} & J_t & \ldots J_{t-s+2}\\ \cdots & \cdots & \cdots\\ J_{t+s-1} & J_{t+s-2} & \ldots J_t\end{vmatrix}$$

and $s \not> t$.

* For the case $m=0$, Theorem VI yields a relation between the total symmetric functions and the elementary products of the a's. If also $s=1$, there results a formula for the total symmetric function H_t in terms of the elementary products, a formula which has already been proved by Roe in the *Transactions of the American Mathematical Society*, Vol. V, No. 2, p. 202, April, 1904.

Proof:

$$\text{Let } \Delta^{\text{II}} \equiv \left|\begin{array}{llllllll}
0 & 0 & \dots & 0 & 1 & 0 & \dots 0 & 0 \\
0 & 0 & \dots & 0 & K_1 & 1 & \dots 0 & 0 \\
\cdot & \cdot & \dots & \cdot & \cdot\cdot & \cdot & \dots\dots & \cdot \\
0 & 0 & \dots & 0 & \cdot\cdot & \cdot & \dots K_1 & 1 \\
(-1)^{t-1}J_t & (-1)^{t-2}J_{t-1} & \dots & (-1)^{t-s}J_{t-s+1} & \cdot\cdot & \cdot & \dots K_2 & K_1 \\
(-1)^{t-1}(K_1J_t-J_{t+1}) & (-1)^{t-2}(K_1J_{t-1}-J_t) & \dots & (-1)^{t-s}(K_1J_{t-s+1}-J_{t-s+2}) & \cdot\cdot & \cdot & \dots K_3 & K_2 \\
\dots\dots & \dots\dots & \dots & \dots\dots & & \cdot & \dots\dots & \cdot\cdot \\
\sum_{t}^{s+t-1}(-1)^{i-1}K_{s+t-1-i}J_i & \sum_{t-1}^{s+t-2}(-1)^{i-1}K_{s+t-2-i}J_i & \dots & \sum_{t-s+1}^{t}(-1)^{i-1}K_{t-i}J_i & K_{t-1} & K_{t-2} & \dots K_{s+1} & K_s
\end{array}\right|,$$

$$\Delta^{\text{III}} \equiv \left|\begin{array}{llll}
(-1)^{t-1}J_t & (-1)^{t-2}J_{t-1} & \dots & (-1)^{t-s}J_{t-s+1} \\
(-1)^{t-1}(K_1J_t-J_{t+1}) & (-1)^{t-2}(K_1J_{t-1}-J_t) & \dots & (-1)^{t-s}(K_1J_{t-s+1}-J_{t-s+2}) \\
\dots\dots & \dots\dots & \dots & \dots\dots \\
\sum_{t}^{s+t-1}(-1)^{i-1}K_{s+t-1-i}J_i & \sum_{t-1}^{s+t-2}(-1)^{i-1}K_{s+t-2-i}J_i & \dots & \sum_{t-s+1}^{t}(-1)^{i-1}K_{t-i}J_i
\end{array}\right|,$$

$$\Delta^{\text{IV}} \equiv \left|\begin{array}{llll}
+J_t & +J_{t-1} & \dots & +J_{t-s+1} \\
-J_{t+1} & -J_t & \dots & -J_{t-s+2} \\
+J_{t+2} & +J_{t+1} & \dots & +J_{t-s+3} \\
\dots & \dots & \dots & \dots \\
(-1)^{s-1}J_{t+s-1} & (-1)^{s-1}J_{t+s-2} & \dots & (-1)^{s-1}J_t
\end{array}\right| \text{ and } D \equiv \left|\begin{array}{llll}
1 & 0 & \dots & 0 \\
K_1 & 1 & \dots & 0 \\
\cdot\cdot & \cdot & \dots & \cdot \\
K_{s-1} & K_{s-2} & \dots & 1
\end{array}\right| \equiv 1.$$

also let $l=(t-s)s$,

$$p=l+\frac{s}{2}(2t-s-1), \quad q=\left\{\begin{array}{l} p+\dfrac{s}{2}, \text{ when } s \text{ is even,} \\[2ex] p+\dfrac{s-1}{2}, \text{ when } s \text{ is odd,} \end{array}\right\}, \ q \text{ therefore}$$

always being even.

If for the K's of the first column of Δ the values given by Theorem III are substituted and the result expressed as the algebraic sum of $s+t-1$ determinants by using the corresponding terms of the elements of the first column for the first columns, then the process repeated for the second, third, ..., s-th column in turn, the sum of the determinants that do not vanish on account of columns differing by a common factor is identical with Δ^{II}. If Δ^{II} is expanded by Laplace's development in terms of its first $t-s$ rows, the result is $(-1)^l \cdot \Delta^{\text{III}}$. After the factor $(-1)^{t-1}$ is removed from the first column, $(-1)^{t-2}$ from the second, and so on, the resulting determinant is, according to the product theorem for determinants, $D \cdot \Delta^{\text{IV}}$. Thus

$$\Delta=(-1)^p \cdot D \cdot \Delta^{\text{IV}}=(-1)^q \cdot \Delta'=\Delta'.$$

Section 3.

$\overline{\begin{pmatrix}1\ldots n\\1\ldots m\end{pmatrix}}^{(m-1)}$ will now be considered. $\begin{pmatrix}1\ldots n\\1\ldots m\end{pmatrix}^{(m-1)}\neq 0$ unless $\begin{pmatrix}1\ldots n\\1\ldots m-1\end{pmatrix}=0$ and $\begin{pmatrix}1\ldots n\\2\ldots m\end{pmatrix}=0$. It vanishes with them except when $\begin{pmatrix}1\ldots n\\2\ldots m-1\end{pmatrix}=0$.* Let $\overline{\begin{pmatrix}1\ldots n\\1\ldots m\end{pmatrix}}^{(m-1)}=\overline{\begin{pmatrix}1\ldots n\\1\ldots m-1\end{pmatrix}}\cdot\overline{\begin{pmatrix}1\ldots n\\2\ldots m\end{pmatrix}}-\overline{X}$. The value of $\overline{X}$ will now be found, or rather a quantity satisfying a more general condition will be found (Theorem IX), and from it the value of $\overline{X}$ will follow as a special case. First, however, some preliminary theorems (VII and VIII) will be proved.

THEOREM VIIa. $\overline{\begin{pmatrix}1\ldots 6\\1\ldots 5\end{pmatrix}}=\overline{\begin{pmatrix}123\\12\end{pmatrix}}+\overline{\begin{pmatrix}12\\12\end{pmatrix}}\cdot\overline{\begin{pmatrix}456\\345\end{pmatrix}}+\overline{\begin{pmatrix}3456\\345\end{pmatrix}}$.

This relation will be proved by a system of sections. The section of $\begin{pmatrix}1\ldots 6\\1\ldots 5\end{pmatrix}=0$ by the spreads $\begin{pmatrix}1\\2\ldots 5\end{pmatrix}=0$ degenerates into two parts of the same dimension for one of which, say A, $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}2\ldots 6\\1\ldots 5\end{pmatrix}$ also vanish, and for the other $B\equiv\begin{pmatrix}2\ldots 6\\2\ldots 5\end{pmatrix}=0$. Thus $\overline{\begin{pmatrix}1\ldots 6\\1\ldots 5\end{pmatrix}}=\overline{A}+\overline{B}$. Again, taking the section of $A=0$ by $\begin{pmatrix}23\\345\end{pmatrix}^{(1)}=0$, the section of $B=0$ by $\begin{pmatrix}2\\345\end{pmatrix}=0$, and then taking the section of one of the parts into which $B=0$ degenerates by $\begin{pmatrix}3\\345\end{pmatrix}=0$, it is found that

$$\overline{\begin{pmatrix}1\ldots 6\\1\ldots 5\end{pmatrix}}=\overline{\begin{pmatrix}1\\1\end{pmatrix}}\left[\overline{\begin{pmatrix}23\\12\end{pmatrix}}+\overline{\begin{pmatrix}456\\345\end{pmatrix}}\right]+\overline{\begin{pmatrix}2\\2\end{pmatrix}}\left[\overline{\begin{pmatrix}3\\2\end{pmatrix}}+\overline{\begin{pmatrix}456\\345\end{pmatrix}}\right]+\overline{\begin{pmatrix}3\ldots 6\\345\end{pmatrix}}$$

$$=\overline{C}+\overline{D}+\overline{E},\ \text{where}\ \overline{C}=\overline{\begin{pmatrix}1\\1\end{pmatrix}}\overline{\begin{pmatrix}23\\12\end{pmatrix}}+\overline{\begin{pmatrix}23\\2\end{pmatrix}},$$

$$\overline{D}=\left[\overline{\begin{pmatrix}1\\1\end{pmatrix}}+\overline{\begin{pmatrix}2\\2\end{pmatrix}}\right]\overline{\begin{pmatrix}456\\345\end{pmatrix}}\ \text{and}\ \overline{E}=\overline{\begin{pmatrix}3\ldots 6\\345\end{pmatrix}}.$$

Now $\overline{C}$ is seen to give the first term required and $\overline{D}$ the second, by taking sections by $\begin{pmatrix}1\\2\end{pmatrix}=0$. The theorem follows.

THEOREM VII. $\overline{\begin{pmatrix}1\ldots n\\1\ldots m\end{pmatrix}}=\sum_{i=n-m+k}^{i=k-1}\overline{\begin{pmatrix}1\ldots i\\1\ldots k\end{pmatrix}}^{(k)}\overline{\begin{pmatrix}i+2\ldots n\\k+1\ldots m\end{pmatrix}}^{(m-k)}$.

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* See author's paper already referred to.

Proof: Theorem VIIa shows the relation to be valid for $n=6$, $m=5$, $k=2$. It will be assumed to be valid for $n-1$, m, k and for $n-1$, $m-1$, $k-1$. Then taking the section of $\begin{pmatrix}1\dots n\\1\dots m\end{pmatrix}=0$ by $\begin{pmatrix}&1&\\2\dots&&m\end{pmatrix}=0$, it follows that

$$\begin{aligned}\begin{pmatrix}\overline{1\dots n}\\1\dots m\end{pmatrix}&=\begin{pmatrix}\overline{1}\\1\end{pmatrix}\begin{pmatrix}\overline{2\dots n}\\1\dots m\end{pmatrix}+\begin{pmatrix}\overline{2\dots n}\\2\dots m\end{pmatrix}\\&=\begin{pmatrix}\overline{1}\\1\end{pmatrix}\sum_{i=n-m+k}^{i=k-1}\begin{pmatrix}\overline{2\dots i}\\1\dots k\end{pmatrix}^{(k)}\begin{pmatrix}\overline{i+2\dots n}\\k+1\dots m\end{pmatrix}^{(m-k)}\\&\qquad+\sum_{i=n-m+k}^{i=k-1}\begin{pmatrix}\overline{2\dots i}\\2\dots k\end{pmatrix}^{(k-1)}\begin{pmatrix}\overline{i+2\dots n}\\k+1\dots m\end{pmatrix}^{(m-k)}.\end{aligned}$$

But $\begin{pmatrix}\overline{1}\\1\end{pmatrix}\begin{pmatrix}\overline{2\dots i}\\1\dots k\end{pmatrix}^{(k)}+\begin{pmatrix}\overline{2\dots i}\\2\dots k\end{pmatrix}^{(k-1)}=\begin{pmatrix}\overline{1\dots i}\\1\dots k\end{pmatrix}^{(k)}$, as may be seen by taking the section of the manifold $\begin{pmatrix}1\dots i\\1\dots k\end{pmatrix}=0$ by $\begin{pmatrix}&1&\\2\dots&&k\end{pmatrix}=0$, and the theorem follows.

Consider the two systems of equations

$$I\equiv\begin{cases}\begin{pmatrix}1\dots k\\1\dots s\end{pmatrix}=0\\[4pt]\begin{pmatrix}j\dots n\\1\dots s\end{pmatrix}^{(s)}=0\end{cases},\quad II\equiv\begin{pmatrix}1\dots n\\1\dots s\end{pmatrix}=0,\quad s\ngtr j\ngtr k\ngtr n.$$

The vanishing of I is a necessary but not sufficient condition for the vanishing of II. For the system I there may be obtained an equivalent system III by means of a formula previously found by the writer—the formula from which $\begin{pmatrix}\overline{1\dots n}\\1\dots m\end{pmatrix}$ is found in Theorem II—and, in general, for a part of this system II vanishes and for a part it does not. Let $Y_{j,k}$ represent the part of the system III, and therefore the part of the system I, for which II vanishes. The value of $\overline{Y}_{j,k}$ will now be investigated.

THEOREM VIIIa. *If* $j>k-s+2$, $\overline{Y}_{j,k}=0$.

For in this case $\begin{pmatrix}1\dots k\\1\dots s\end{pmatrix}$ and $\begin{pmatrix}j\dots n\\1\dots s\end{pmatrix}^{(s)}$ combined contain not more than $[k-s+1]+[n-j+1)-s+1]$ linearly independent determinants, which is less than $n-s+1$, the number contained in $\begin{pmatrix}1\dots n\\1\dots s\end{pmatrix}$ and $\begin{pmatrix}1\dots n\\1\dots s\end{pmatrix}$ can not, in general, be made to vanish with $\begin{pmatrix}1\dots k\\1\dots s\end{pmatrix}$ and $\begin{pmatrix}j\dots n\\1\dots s\end{pmatrix}^{(s)}$.

THEOREM VIIIb. *If* $j=k-s+2$ *and* $k=n-1$, *then* $\overline{Y}_{j,k}=\overline{Y}_{n-s+1,\,n-1}$ $=\binom{1\dots n}{1\dots s}$.

Proof: $\binom{\overline{1\dots n}}{1\dots s}=\sum_1^s(-1)^{i+1}\binom{\overline{1\dots n-i}}{1\dots s}^{(s)}\binom{\overline{n-s+1\dots n-i+1}}{1\dots s}^{(s-i+1)}$, by Theorem II. Therefore $\binom{\overline{1\dots n-1}}{1\dots s}^{(s)}\binom{\overline{n-s+1\dots n}}{1\dots s}^{(s)}=\binom{\overline{1\dots n}}{1\dots s}^{(s)}+\overline{F}$, where none of the matrices in F contain the n-th column and for none of them, therefore, does $\binom{1\dots n}{1\dots s}$ vanish. Therefore $\overline{Y}_{n-s+1,\,n-1}=\binom{\overline{1\dots n}}{1\dots s}^{(s)}$.

THEOREM VIIIc. *If* $j=k-s+2$, $\overline{Y}_{j,k}=\overline{Y}_{k-s+2,k}=\binom{\overline{1\dots n}}{1\dots s}$.

Proof: By using the value of $\binom{\overline{1\dots n}}{1\dots s}$ of Theorem VIIIb and reducing $\binom{\overline{1\dots n-1}}{1\dots s}$ by means of Theorem II

$$\begin{aligned}\binom{\overline{1\dots n}}{1\dots s}&=\binom{\overline{n-s+1\dots n}}{1\dots s}\left[\sum_1^s(-1)^{i+1}\binom{\overline{1\dots n-i-1}}{1\dots s}^{(s)}\binom{\overline{n-s\dots n-i}}{1\dots s}^{(s-i+1)}\right]\\&\quad+\sum_1^{s-1}(-1)^i\binom{\overline{1\dots n-i-1}}{1\dots s}^{(s)}\binom{\overline{n-s+1\dots n-i}}{1\dots s}^{(s-i)}\\&=\binom{\overline{1\dots n-2}}{1\dots s}^{(s)}\left[\binom{\overline{n-s\dots n-1}}{1\dots s}^{(s)}\binom{\overline{n-s+1\dots n}}{1\dots s}^{(s)}-\binom{\overline{n-s+1\dots n-1}}{1\dots s}^{(s-1)}\right]\\&\quad+\binom{\overline{1\dots n-3}}{1\dots s}^{(s)}\left[-\binom{\overline{n-s\dots n-2}}{1\dots s}^{(s-1)}\binom{\overline{n-s+1\dots n}}{1\dots s}^{(s)}\right.\\&\quad\left.+\binom{\overline{n-s+1\dots n-2}}{1\dots s}^{(s-2)}\right]+\binom{\overline{1\dots n-4}}{1\dots s}^{(s)}[\;]+\dots\\&=\binom{\overline{1\dots n-2}}{1\dots s}^{(s)}\binom{\overline{n-s\dots n}}{1\dots s}^{(s)}+\binom{\overline{1\dots n-3}}{1\dots s}^{(s)}\left[\binom{\overline{n-s-1\dots n}}{1\dots s}^{(s)}\right.\\&\quad\left.-\binom{\overline{n-s-1\dots n-2}}{1\dots s}^{(s)}\binom{\overline{n-s\dots n}}{1\dots s}^{(s)}\right]+\binom{\overline{1\dots n-4}}{1\dots s}^{(s)}[\;]+\dots.\end{aligned}$$

We will suppose this relation to be valid in the formula obtained for $\binom{\overline{1\dots n}}{1\dots s}^{(s)}$ after $n-k-2$ reductions have been made, that is, that

$$\begin{aligned}\binom{\overline{1\dots n}}{1\dots s}&=\binom{\overline{1\dots k+1}}{1\dots s}^{(s)}\binom{\overline{k-s+3\dots n}}{1\dots s}^{(s)}+\binom{\overline{1\dots k}}{1\dots s}^{(s)}\left[\binom{\overline{k-s+2\dots n}}{1\dots s}^{(s)}\right.\\&\quad\left.-\binom{\overline{k-s+2\dots k+1}}{1\dots s}^{(s)}\binom{\overline{k-s+3\dots n}}{1\dots s}^{(s)}\right]+\binom{\overline{1\dots k-1}}{1\dots s}^{(s)}[\;]+\dots.\end{aligned}$$

Reducing $\begin{pmatrix}\overline{1\ldots k+1}\\1\ldots s\end{pmatrix}^{(s)}$ by Theorem II,

$$\begin{aligned}\begin{pmatrix}\overline{1\ldots n}\\1\ldots s\end{pmatrix}^{(s)}=\begin{pmatrix}\overline{k-s+3\ldots n}\\1\ldots s\end{pmatrix}^{(s)}\Bigg[&\begin{pmatrix}\overline{1\ldots k}\\1\ldots s\end{pmatrix}^{(s)}\begin{pmatrix}\overline{k-s+2\ldots k+1}\\1\ldots s\end{pmatrix}^{(s)}\\&-\begin{pmatrix}\overline{1\ldots k-1}\\1\ldots s\end{pmatrix}^{(s)}\begin{pmatrix}\overline{k-s+2\ldots k}\\1\ldots s\end{pmatrix}^{(s-1)}+\ldots\Bigg]\\&+\begin{pmatrix}\overline{1\ldots k}\\1\ldots s\end{pmatrix}^{(s)}\Bigg[\begin{pmatrix}\overline{k-s+2\ldots n}\\1\ldots s\end{pmatrix}^{(s)}\\&-\begin{pmatrix}\overline{k-s+2\ldots k+1}\\1\ldots s\end{pmatrix}^{(s)}\begin{pmatrix}\overline{k-s+3\ldots n}\\1\ldots s\end{pmatrix}^{(s)}\Bigg]+\ldots\\&=\begin{pmatrix}\overline{1\ldots k}\\1\ldots s\end{pmatrix}^{(s)}\begin{pmatrix}\overline{k-s+2\ldots n}\\1\ldots s\end{pmatrix}^{(s)}+\begin{pmatrix}\overline{1\ldots k-1}\\1\ldots s\end{pmatrix}^{(s)}[\;]+\ldots.\end{aligned}$$

Hence $\begin{pmatrix}\overline{1\ldots k}\\1\ldots s\end{pmatrix}^{(s)}\begin{pmatrix}\overline{k-s+2\ldots n}\\1\ldots s\end{pmatrix}^{(s)}=\begin{pmatrix}\overline{1\ldots n}\\1\ldots s\end{pmatrix}^{(s)}+\overline{F}$, where none of the matrices in F contain the n-th column. Therefore $Y_{j,k}=\overline{Y}_{k-s+2,k}=\begin{pmatrix}\overline{1\ldots n}\\1\ldots s\end{pmatrix}^{(s)}$.

THEOREM VIIId. *If* $j=k-s+1$,

$$\overline{Y}_{j,k}=\overline{Y}_{k-s+1,k}=\begin{pmatrix}\overline{k-s+1\ldots n}\\1\ldots s\end{pmatrix}^{(s)}\begin{pmatrix}\overline{1\ldots n}\\1\ldots s\end{pmatrix}^{(s)}.$$

Proof:

$$\begin{aligned}\begin{pmatrix}\overline{1\ldots k}\\1\ldots s\end{pmatrix}^{(s)}\begin{pmatrix}\overline{k-s+1\ldots n}\\1\ldots s\end{pmatrix}^{(s)}=\begin{pmatrix}\overline{1\ldots k}\\1\ldots s\end{pmatrix}\Bigg[&\begin{pmatrix}\overline{k-s+2\ldots n}\\1\ldots s\end{pmatrix}^{(s)}\begin{pmatrix}\overline{k-s+1\ldots k}\\1\ldots s\end{pmatrix}^{(s)}\\&-\begin{pmatrix}\overline{k-s+3\ldots n}\\1\ldots s\end{pmatrix}^{(s)}\begin{pmatrix}\overline{k-s+2\ldots k}\\1\ldots s\end{pmatrix}^{(s-1)}\\&+\begin{pmatrix}\overline{k-s+4\ldots n}\\1\ldots s\end{pmatrix}^{(s)}\begin{pmatrix}\overline{k-s+3\ldots k}\\1\ldots s\end{pmatrix}^{(s-2)}-\ldots\Bigg].\end{aligned}$$

Therefore

$$\begin{aligned}Y_{k-s+1,k}=&\begin{pmatrix}\overline{k-s+1\ldots k}\\1\ldots s\end{pmatrix}^{(s)}\overline{Y}_{k-s+2,k}\\&-\begin{pmatrix}\overline{k-s+2\ldots k}\\1\ldots s\end{pmatrix}^{(s-1)}\overline{Y}_{k-s+3,k}+\begin{pmatrix}\overline{k-s+3\ldots k}\\1\ldots s\end{pmatrix}^{(s-2)}\overline{Y}_{k-s+4,k}-\ldots.\end{aligned}$$

But $\overline{Y}_{k-s+2,k}=\begin{pmatrix}\overline{1\ldots n}\\1\ldots s\end{pmatrix}^{(s)}$ by Theorem VIIIc and $\overline{Y}_{k-s+3,k}=\overline{Y}_{k-s+4,k}=\ldots=0$ by Theorem VIIIa. Therefore

$$\overline{Y}_{j,k}=\overline{Y}_{k-s+1,k}=\begin{pmatrix}\overline{k-s+1\ldots k}\\1\ldots s\end{pmatrix}^{(s)}\begin{pmatrix}\overline{1\ldots n}\\1\ldots s\end{pmatrix}^{(s)}.$$

THEOREM VIII. *The order* $\overline{Y}_{j,k}$ *of the part of the system* $\begin{pmatrix}1\dots k\\1\dots s\end{pmatrix}=0$, $\begin{pmatrix}j\dots n\\1\dots s\end{pmatrix}^{(s)}=0$ *for which* $\begin{pmatrix}1\dots n\\1\dots s\end{pmatrix}^{(s)}=0$ *is*

$$\overline{\begin{pmatrix}j\dots k\\1\dots s\end{pmatrix}}^{(s)}\overline{\begin{pmatrix}1\dots n\\1\dots s\end{pmatrix}}^{(s)},\quad s \not> j \not> k \not> n.$$

Proof: Suppose it was already shown that

$$\overline{Y}_{j+i,k}=\overline{\begin{pmatrix}j+i\dots k\\1\dots s\end{pmatrix}}^{(s)}\overline{\begin{pmatrix}1\dots n\\1\dots s\end{pmatrix}}^{(s)},\quad i=1\dots k-j.$$

Since $\overline{\begin{pmatrix}1\dots k\\1\dots s\end{pmatrix}}^{(s)}\overline{\begin{pmatrix}j\dots n\\1\dots s\end{pmatrix}}^{(s)}$

$$=\overline{\begin{pmatrix}1\dots k\\1\dots s\end{pmatrix}}^{(s)}\sum_{i=1}^{i=s}(-1)^{i+1}\overline{\begin{pmatrix}j+i\dots n\\1\dots s\end{pmatrix}}^{(s)}\overline{\begin{pmatrix}j-1+i\dots j+s-1\\1\dots s\end{pmatrix}}^{(s-i+1)},$$

by Theorem II,

$$=\sum_{i=1}^{i=s}(-1)^{i+1}\overline{\begin{pmatrix}j-1+i\dots j+s-1\\1\dots s\end{pmatrix}}^{(s-i+1)}\left[\overline{\begin{pmatrix}1\dots k\\1\dots s\end{pmatrix}}^{(s)}\overline{\begin{pmatrix}j+i\dots n\\1\dots s\end{pmatrix}}^{(s)}\right].$$

Therefore

$$\overline{Y}_{j,k}=\sum_{i=1}^{i=s}(-1)^{i+1}\overline{\begin{pmatrix}j-1+i\dots j+s-1\\1\dots s\end{pmatrix}}^{(s-i+1)}\overline{Y}_{j+i,k}$$

$$=\sum_{i=1}^{i=s}(-1)^{i+1}\overline{\begin{pmatrix}j-1+i\dots j+s-1\\1\dots s\end{pmatrix}}^{(s-i+1)}\overline{\begin{pmatrix}j+i\dots k\\1\dots s\end{pmatrix}}^{(s)}\overline{\begin{pmatrix}1\dots n\\1\dots s\end{pmatrix}}^{(s)}$$

$$=\overline{\begin{pmatrix}1\dots n\\1\dots s\end{pmatrix}}^{(s)}\sum_{i=1}^{i=s}(-1)^{i+1}\overline{\begin{pmatrix}j+i\dots k\\1\dots s\end{pmatrix}}^{(s)}\overline{\begin{pmatrix}j-1+i\dots j+s-1\\1\dots s\end{pmatrix}}^{(s-i+1)}$$

$$=\overline{\begin{pmatrix}j\dots k\\1\dots s\end{pmatrix}}^{(s)}\overline{\begin{pmatrix}1\dots n\\1\dots s\end{pmatrix}}^{(s)},\ \text{by Theorem II.}$$

It is to be noticed that Theorems VIIIa, b, c are special cases of Theorem VIII, since by definition of the order symbol $\overline{\begin{pmatrix}j\dots k\\1\dots s\end{pmatrix}}^{(s)}$ has the value 1 when $j=k-s+2$, and the value 0 when $j>k-s+2$.

THEOREM IX. *The order* $\overline{X}_{h,l}$ *of the part of the system* $\begin{pmatrix}1\dots n\\1\dots l\end{pmatrix}^{(l)}=0$, $\begin{pmatrix}1\dots n\\h\dots m\end{pmatrix}^{(m-h+1)}=0$, *for which* $\begin{pmatrix}1\dots n\\h\dots l\end{pmatrix}^{(l-h+1)}=0$ *is*

$$\overline{\begin{pmatrix}1\dots n\\h\dots l\end{pmatrix}}^{(l-h+1)}\overline{\begin{pmatrix}1\dots n\\1\dots m\end{pmatrix}}^{(m)},\quad h \not> l \not> m \not> n.$$

Proof: $\begin{pmatrix}\overline{1\ldots n}\\1\ldots l\end{pmatrix}^{(l)}=\sum\limits_{i=n-l+h-1}^{i=h-2}\begin{pmatrix}\overline{1\ldots i}\\1\ldots h-1\end{pmatrix}^{(h-1)}\begin{pmatrix}\overline{i+2\ldots n}\\h\ldots l\end{pmatrix}^{(l-h+1)}$

and $\begin{pmatrix}\overline{1\ldots n}\\h\ldots m\end{pmatrix}^{(m-h+1)}=\sum\limits_{j=n-m+l}^{j=l-h}\begin{pmatrix}\overline{1\ldots j}\\h\ldots l\end{pmatrix}^{(l-h+1)}\begin{pmatrix}\overline{j+2\ldots n}\\l+1\ldots m\end{pmatrix}^{(m-l)}$,

according to Theorem VII. Therefore

$$\begin{aligned}
&\begin{pmatrix}\overline{1\ldots n}\\1\ldots l\end{pmatrix}^{(l)}\begin{pmatrix}\overline{1\ldots n}\\h\ldots m\end{pmatrix}^{(m-h+1)}\\
&=\sum_{i=n-l+h-1}^{i=h-2}\begin{pmatrix}\overline{1\ldots i}\\1\ldots h-1\end{pmatrix}^{(h-1)}\begin{pmatrix}\overline{i+2\ldots n}\\h\ldots l\end{pmatrix}^{(l-h+1)}\sum_{j=n-m+l}^{j=l-h}\begin{pmatrix}\overline{1\ldots j}\\h\ldots l\end{pmatrix}^{(l-h+1)}\begin{pmatrix}\overline{j+2\ldots n}\\l+1\ldots m\end{pmatrix}^{(m-l)}\\
&=\sum_{i=j-l+h-1}^{i=h-2}\begin{pmatrix}\overline{1\ldots i}\\1\ldots h-1\end{pmatrix}^{(h-1)}\cdot\sum_{j=n-m+l}^{j=l-h}\begin{pmatrix}\overline{1\ldots j}\\h\ldots l\end{pmatrix}^{(l-h+1)}\begin{pmatrix}\overline{i+2\ldots n}\\h\ldots l\end{pmatrix}^{(l-h+1)}\begin{pmatrix}\overline{j+2\ldots n}\\l+1\ldots m\end{pmatrix}^{(m-l)}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\overline{X}_{h,l}&=\sum_{i=j-l+h-1}^{i=h-2}\begin{pmatrix}\overline{1\ldots i}\\1\ldots h-1\end{pmatrix}^{(h-1)}\cdot\sum_{j=n-m+l}^{j=l-h}\overline{Y}_{i+2,j}\begin{pmatrix}\overline{j+2\ldots n}\\l+1\ldots m\end{pmatrix}^{(m-l)}\\
&=\sum_{i=j-l+h-1}^{i=h-2}\begin{pmatrix}\overline{1\ldots i}\\1\ldots h-1\end{pmatrix}^{(h-1)}\cdot\sum_{j=n-m+l}^{j=l-h}\begin{pmatrix}\overline{i+2\ldots j}\\h\ldots l\end{pmatrix}^{(l-h+1)}\begin{pmatrix}\overline{1\ldots n}\\h\ldots l\end{pmatrix}^{(l-h+1)}\begin{pmatrix}\overline{j+2\ldots n}\\l+1\ldots m\end{pmatrix}^{(m-l)},
\end{aligned}$$

by Theorem VIII,

$$\begin{aligned}
&=\begin{pmatrix}\overline{1\ldots n}\\h\ldots l\end{pmatrix}^{(l-h+1)}\sum_{i=j-l+h-1}^{i=h-2}\begin{pmatrix}\overline{1\ldots i}\\1\ldots h-1\end{pmatrix}^{(h-1)}\cdot\sum_{j=n-m+l}^{j=l-h}\begin{pmatrix}\overline{i+2\ldots j}\\h\ldots l\end{pmatrix}^{(l-h+1)}\begin{pmatrix}\overline{j+2\ldots n}\\l+1\ldots m\end{pmatrix}^{(m-l)}\\
&=\begin{pmatrix}\overline{1\ldots n}\\h\ldots l\end{pmatrix}^{(l-h+1)}\sum_{i=j-l+h-1}^{i=h-2}\begin{pmatrix}\overline{1\ldots i}\\1\ldots h-1\end{pmatrix}^{(h-1)}\begin{pmatrix}\overline{i+2\ldots n}\\h\ldots m\end{pmatrix}^{(m-h+1)},\text{ by Theorem VII,}\\
&=\begin{pmatrix}\overline{1\ldots n}\\h\ldots l\end{pmatrix}^{(l-h+1)}\begin{pmatrix}\overline{1\ldots n}\\1\ldots m\end{pmatrix}^{(m)},\text{ by Theorem VII.}
\end{aligned}$$

COROLLARY: $\overline{X}_{h,k}=\phi\begin{pmatrix}\overline{1\ldots n}\\1\ldots l\end{pmatrix}\begin{pmatrix}\overline{1\ldots n}\\h\ldots m\end{pmatrix}$, *according to the definition of* ϕ.

Section 4.

THEOREM X. *If* $m \not> n$, $\begin{pmatrix}\overline{1\ldots n}\\1\ldots m\end{pmatrix}^{(m-1)}=\begin{vmatrix}\begin{pmatrix}\overline{1\ldots n}\\1\ldots m-1\end{pmatrix} & \begin{pmatrix}\overline{1\ldots n}\\1\ldots m\end{pmatrix}\\ \begin{pmatrix}\overline{1\ldots n}\\2\ldots m-1\end{pmatrix} & \begin{pmatrix}\overline{1\ldots n}\\2\ldots m\end{pmatrix}\end{vmatrix}$.

Proof: As previously pointed out $\begin{pmatrix}\overline{1\ldots n}\\1\ldots m\end{pmatrix}^{(m-1)}=\begin{pmatrix}\overline{1\ldots n}\\1\ldots m-1\end{pmatrix}\begin{pmatrix}\overline{1\ldots n}\\2\ldots m\end{pmatrix}$ $-\overline{X}_{2,m-1}=\begin{pmatrix}\overline{1\ldots n}\\1\ldots m-1\end{pmatrix}\begin{pmatrix}\overline{1\ldots n}\\2\ldots m\end{pmatrix}-\begin{pmatrix}\overline{1\ldots n}\\1\ldots m\end{pmatrix}\begin{pmatrix}\overline{1\ldots n}\\2\ldots m-1\end{pmatrix}$, by Theorem IX. The theorem follows.

COROLLARY: $\left(\overline{\begin{matrix}1\ldots.n\\1\ldots.\mathrm{m}\end{matrix}}\right)^{(m-1)}=\begin{vmatrix}K'_2 & K'_1\\ K'_3 & K'_2\end{vmatrix}$, *where* $K'_i=K_{n-m+i}$, $m \not> n$.

Proof: $\left(\overline{\begin{matrix}1\ldots.n\\1\ldots.m\end{matrix}}\right)^{(m-1)}=\begin{vmatrix}\underset{a=1\ldots.m-1}{K'_2} & \underset{a=1\ldots.m}{K'_1}\\ \underset{a=2\ldots.m-1}{K'_3} & \underset{a=2\ldots.m}{K'_2}\end{vmatrix}$, by theorems X and I,

$$=\begin{vmatrix}\underset{a=1\ldots.m}{K'_2-a_mK'_1} & \underset{a=1\ldots.m}{K'_1}\\ \underset{a=2\ldots.m}{K'_3-a_mK'_2} & \underset{a=2\ldots.m}{K'_2}\end{vmatrix}, \text{ by Theorem IV,}$$

$$=\begin{vmatrix}\underset{a=1\ldots.m}{K'_2} & \underset{a=1\ldots.m}{K'_1}\\ \underset{a=2\ldots.m}{K'_3} & \underset{a=2\ldots.m}{K'_2}\end{vmatrix}.$$

By introducing a_1 into the second row in a similar way, the corollary follows.

THEOREM XI. *If* $m \not> n \left(\overline{\begin{matrix}1\ldots.n\\1\ldots.m\end{matrix}}\right)^{(m-2)}$

$$=\begin{vmatrix}\left(\overline{\begin{matrix}1\ldots.n\\1\ldots.m-2\end{matrix}}\right) & \left(\overline{\begin{matrix}1\ldots.n\\1\ldots.m-1\end{matrix}}\right) & \left(\overline{\begin{matrix}1\ldots.n\\1\ldots.m\end{matrix}}\right)\\ \left(\overline{\begin{matrix}1\ldots.n\\2\ldots.m-2\end{matrix}}\right) & \left(\overline{\begin{matrix}1\ldots.n\\2\ldots.m-1\end{matrix}}\right) & \left(\overline{\begin{matrix}1\ldots.n\\2\ldots.m\end{matrix}}\right)\\ \left(\overline{\begin{matrix}1\ldots.n\\3\ldots.m-2\end{matrix}}\right) & \left(\overline{\begin{matrix}1\ldots.n\\3\ldots.m-1\end{matrix}}\right) & \left(\overline{\begin{matrix}1\ldots.n\\3\ldots.m\end{matrix}}\right)\end{vmatrix} = say\ \Delta_3.$$

Proof: $\left(\begin{matrix}1\ldots.n\\1\ldots.m\end{matrix}\right)^{(m-2)} \neq 0$ unless $\left(\begin{matrix}1\ldots.n\\1\ldots.m-1\end{matrix}\right)^{(m-2)}=0$ and $\left(\begin{matrix}1\ldots.n\\3\ldots.m\end{matrix}\right)=0$.

It vanishes with them except when $\left(\begin{matrix}1\ldots.n\\3\ldots.m-1\end{matrix}\right)=0.$*

Hence $\left(\overline{\begin{matrix}1\ldots.n\\1\ldots.m\end{matrix}}\right)^{(m-2)}=\left(\overline{\begin{matrix}1\ldots.n\\1\ldots.m-1\end{matrix}}\right)^{(m-2)}\left(\overline{\begin{matrix}1\ldots.n\\3\ldots.m\end{matrix}}\right)-\bar{X}_{3,m-1}$

$$=(1-\phi)\left(\overline{\begin{matrix}1\ldots.n\\3\ldots.m\end{matrix}}\right)\left(\overline{\begin{matrix}1\ldots.n\\1\ldots.m-1\end{matrix}}\right)^{(m-2)},$$

by Theorem IX, corollary,

$$=(1-\phi)\left(\overline{\begin{matrix}1\ldots.n\\3\ldots.m\end{matrix}}\right)\begin{vmatrix}\left(\overline{\begin{matrix}1\ldots.n\\1\ldots.m-1\end{matrix}}\right) & \left(\overline{\begin{matrix}1\ldots.n\\1\ldots.m\end{matrix}}\right)\\ \left(\overline{\begin{matrix}1\ldots.n\\2\ldots.m-1\end{matrix}}\right) & \left(\overline{\begin{matrix}1\ldots.n\\2\ldots.m\end{matrix}}\right)\end{vmatrix},$$

by Theorem X,

$=\Delta_3$, by Theorem V.

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*See author's paper already referred to.

COROLLARY: $\binom{\overline{1\ldots n}}{1\ldots m}^{(m-2)} = \begin{vmatrix} K'_3 & K'_2 & K'_1 \\ K'_4 & K'_3 & K'_2 \\ K'_5 & K'_4 & K'_3 \end{vmatrix}, \; K'_i = K_{n-m+i}, \quad m \not> n.$

Proof: $\binom{\overline{1\ldots n}}{1\ldots m}^{(m-2)} = \begin{vmatrix} \underset{a=1\ldots m-2}{K'_3} & \underset{a=1\ldots m-1}{K'_2} & \underset{a=1\ldots m}{K'_1} \\ \underset{a=2\ldots m-2}{K'_4} & \underset{a=2\ldots m-1}{K'_3} & \underset{a=2\ldots m}{K'_2} \\ \underset{a=3\ldots m-2}{K'_5} & \underset{a=3\ldots m-1}{K'_4} & \underset{a=3\ldots m}{K'_3} \end{vmatrix}$, by Theorems XI and I.

By means of Theorem IV the missing α's may be introduced as in the proof of the corollary of Theorem X, completing the proof.

THEOREM XII. *If* $m \not> n$,

$$\binom{\overline{1\ldots n}}{1\ldots m}^{(r)} = \begin{vmatrix} \binom{\overline{1\ldots n}}{1\ldots r} & \binom{\overline{1\ldots n}}{1\ldots r+1} & \ldots\binom{\overline{1\ldots n}}{1\ldots m} \\ \binom{\overline{1\ldots n}}{2\ldots r} & \binom{\overline{1\ldots n}}{2\ldots r+1} & \ldots\binom{\overline{1\ldots n}}{2\ldots m} \\ \ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots\ldots \\ \binom{\overline{1\ldots n}}{m-r+1\ldots r} & \binom{\overline{1\ldots n}}{m-r+1\ldots r+1} & \ldots\binom{\overline{1\ldots n}}{m-r+1\ldots m} \end{vmatrix}$$

$= say\ \Delta_{m-r+1}$.

Proof: It will be assumed that a relation shown in Theorem X to be valid for $\binom{\overline{1\ldots n}}{1\ldots m}^{(m-1)}$ is valid for $\binom{\overline{1\ldots n}}{1\ldots m-1}^{(r)}$, namely:

$$\binom{\overline{1\ldots n}}{1\ldots m-1}^{(r)} = \begin{vmatrix} \binom{\overline{1\ldots n}}{1\ldots r} & \binom{\overline{1\ldots n}}{1\ldots r+1} & \ldots\binom{\overline{1\ldots n}}{1\ldots m-1} \\ \binom{\overline{1\ldots n}}{2\ldots r} & \binom{\overline{1\ldots n}}{2\ldots r+1} & \ldots\binom{\overline{1\ldots n}}{2\ldots m-1} \\ \ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots\ldots \\ \binom{\overline{1\ldots n}}{m-r\ldots r} & \binom{\overline{1\ldots n}}{m-r\ldots r+1} & \ldots\binom{\overline{1\ldots n}}{m-r\ldots m-1} \end{vmatrix}$$

$=$ say Δ_{m-r}.

Then $\binom{\overline{1\ldots n}}{1\ldots m}^{(r)} = \binom{\overline{1\ldots n}}{m-r+1\ldots m}\binom{\overline{1\ldots n}}{1\ldots m-1}^{(r)} - \bar{\bar{X}}_{m-r+1,\,m-1}$

$= (1-\phi)\binom{\overline{1\ldots n}}{m-r+1\ldots m}\binom{\overline{1\ldots n}}{1\ldots m-1}^{(r)}$,

by Theorem IX, corollary,

$= (1-\phi)\binom{\overline{1\ldots n}}{m-r+1\ldots m}\Delta_{m-r} = \Delta_{m-r+1}$, by Theorem V.

COROLLARY: $$\binom{\overline{1\dots n}}{1\dots m}^{(r)}=\begin{vmatrix} K'_{m-r+1} & K'_{m-r} & \dots K'_1 \\ K'_{m-r+2} & K'_{m-r+1} & \dots K'_2 \\ \dots & \dots & \dots \\ K'_{2(m-r)+1} & K'_{2(m-r)} & \dots K'_{m-r+1} \end{vmatrix}$$
where $K'_i=K_{n-m+i}$, $m \not> n$.

Proof: $$\binom{\overline{1\dots n}}{1\dots m}^{(r)}=\begin{vmatrix} \underset{\alpha=1\dots r}{K'_{m-r+1}} & \underset{\alpha=1\dots r+1}{K'_{m-r}} & \underset{\alpha=1\dots m}{K'_1} \\ \underset{\alpha=2\dots r}{K'_{m-r+2}} & \underset{\alpha=2\dots r+1}{K'_{m-r+1}} & \underset{\alpha=2\dots m}{K'_2} \\ \dots & \dots & \dots \\ \underset{\alpha=m-r+1\dots r}{K'_{2(m-r)+1}} & \underset{\alpha=m-r+1\dots r+1}{K'_{2(m-r)}} & \underset{\alpha=m-r+1\dots m}{K'_{m-r+1}} \end{vmatrix},$$
by Theorems XII and I.

By means of Theorem IV the missing α's may be introduced as in the proof of the corollary of Theorem X, completing the proof.

THEOREM XIII. *For all values of* n,

$$\binom{\overline{1\dots n}}{1\dots m}^{(r)}=\begin{vmatrix} K_{n-r+1} & K_{n-r} & \dots K_{n-m+1} \\ K_{n-r+2} & K_{n-r+1} & \dots K_{n-m+2} \\ \dots & \dots & \dots \\ K_{n-2r+m+1} & K_{n-2r+m} & \dots K_{n-r+1} \end{vmatrix}. \qquad \text{(A)}$$

Proof: Theorem XII, corollary gives the result if $m \not> n$. If $m>n$, $\binom{\overline{1\dots n}}{1\dots m}^{(r)}$ can be calculated by the method of this paper with the interchange of rows and columns. The result will be the Δ' of Theorem VI where $t=m-r+1$ and $s=n-r+1$. But when $m>n$ the determinant of Theorem XII, corollary becomes the Δ of Theorem VI with the same values of t and s. The theorem follows.

COROLLARY: *For all values of* n,

$$\binom{\overline{1\dots n}}{1\dots m}^{(r)}=\begin{vmatrix} \binom{\overline{1\dots n}}{1\dots r} & \binom{\overline{1\dots n}}{1\dots r+1} & \dots \binom{\overline{1\dots n}}{1\dots m} \\ \binom{\overline{1\dots n}}{2\dots r} & \binom{\overline{1\dots n}}{2\dots r+1} & \dots \binom{\overline{1\dots n}}{2\dots m} \\ \dots & \dots & \dots \\ \binom{\overline{1\dots n}}{m-r+1\dots r} & \binom{\overline{1\dots n}}{m-r+1\dots r+1} & \dots \binom{\overline{1\dots n}}{m-r+1\dots m} \end{vmatrix}.$$

This follows from Theorem XIII by retracing the steps of the proof of Theorem XII, corollary.

Section 5.

THEOREM XIV. $\overline{\begin{pmatrix}1 \dots n\\ 1\dots m\end{pmatrix}}^{(r)}$, *where each element u_{ij} is a function of degree b is* $\dfrac{\prod\limits_{i=0}^{i=m-r} {}_{n+m-r}C_{r-1+i}}{\prod\limits_{i=0}^{i=m-r} {}_{n+m-r}C_i} \cdot b^{(m-r+1)(n-r+1)}$

Proof: In this case the a's may be taken each equal to b, and the α's each equal to zero. K_i then reduces to $[\sum\limits_1^n a_1 a_2 \dots a_i]_{a_j=b} = {}_nC_r b^i$, and $\overline{\begin{pmatrix}1 \dots n\\ 1\dots m\end{pmatrix}}^{(r)}$ to

$$b^{(m-r+1)(n-r+1)} \cdot \Delta \text{ where } \Delta = \begin{vmatrix} {}_nC_{n-r+1} & {}_nC_{n-r} & \dots & {}_nC_{n-m+1} \\ {}_nC_{n-r+2} & {}_nC_{n-r+1} & \dots & {}_nC_{n-m+2} \\ \dots & \dots & \dots & \dots \\ {}_nC_{n-2r+m+1} & {}_nC_{n-2r+m} & \dots & {}_nC_{n-r+1} \end{vmatrix}.$$

Next we apply the formula ${}_nC_i = {}_nC_{n-i}$ to each of the elements and multiply the resulting determinant by unity in the form of the determinant

$$\begin{vmatrix} {}_{m-r}C_0 & 0 & \dots & 0 \\ {}_{m-r}C_1 & {}_{m-r-1}C_0 & \dots & 0 \\ \dots & \dots & \dots & . \\ {}_{m-r}C_{m-r} & {}_{m-r-1}C_{m-r-1} & \dots & 1 \end{vmatrix}.$$

This gives for the element in the h-th row and the k-th column,

$${}_{n+m-r+1-h}C_{r-1-h+k} = \frac{(n+m-r+1-h)!}{(r-1-h+k)!(n+m-2r+2-k)!}$$
$$= \frac{(n+m-r+1-h)!\,{}_{r-2+k}P_{h-1}}{(r-2+k)!(n+m-2r+2-k)!},\ {}_lP_0 = 1.$$

The factorial number in the numerator is constant for a given row as is the denominator for a given column. Hence $\Delta = A\Delta'$, where

$$A = \frac{\sum\limits_{h=1}^{h=m-r+1} (n+m-r+1-h)!}{\sum\limits_{k=1}^{k=m-r+1} (r-2+k)!(n+m-2r+2-k)!} \text{ and } \Delta' = \begin{vmatrix} h=1 \dots m-r+1 \\ {}_{r-2+k}P_{h-1} \\ k=1 \dots m-r+1 \end{vmatrix} = |a_{h,k}|.$$

Let $\underset{k=2\dots m-r+1}{D'a_{h,k}} \equiv a_{h,k} - a_{h,k-1}$, $D'a_{h,1} = a_{h,1}$. Then $|D'a_{h,k}| \equiv \Delta'$.

$$D'a_{h,k} = {}_{r-2+k}P_{h-1} - {}_{r-3+k}P_{h-1} = [(r-2+k)-(r-1+k-h)]_{r-3+k}P_{h-2}$$
$$= (h-1)_{r-3+k}P_{h-2},$$

. . . .

Finally

$$D^{(k-1)}a_{h,k} \equiv D^{(k-2)}a_{h,k} - D^{(k-2)}a_{h,k-1} = (h-1)(h-2)\ldots(h-k+1)_{r-1}P_{h-k}$$
$$= \begin{cases} (h-1)!, & \text{if } k=h \\ 0, & \text{if } k>h. \end{cases}$$

Hence
$$\Delta' \equiv \begin{vmatrix} 0! & 0 & 0 & 0 & \ldots & 0 \\ & 1! & 0 & 0 & \ldots & 0 \\ & & 2! & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & . \\ \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & (m-r)! \end{vmatrix} = \prod_{i=0}^{i=m-r} i!.$$

Hence
$$\Delta = A'\Delta = \frac{\prod\limits_{i=0}^{i=m-r}(n+m-r-i)!\ \prod\limits_{i=0}^{i=m-r} i!}{\prod\limits_{i=0}^{i=m-r}(r+i-1)!\ \prod\limits_{i=0}^{i=m-r}(n+m-2r+1-i)!} = \frac{\prod\limits_{i=0}^{i=m-r} {}_{n+m-r}C_{r-1+i}}{\prod\limits_{i=0}^{i=m-r} {}_{n+m-r}C_i}.$$

The theorem follows.

COROLLARY: $\begin{pmatrix}\overline{1\ldots n} \\ 1\ldots m\end{pmatrix}^{(r)}$, *where each element* u_{ij} *is a function of degree* b, *is*

$$\frac{\prod\limits_{i=0}^{i=m-r} {}_{n+i}C_{n-r+1}}{\prod\limits_{i=0}^{i=m-r} {}_{n-r+1+i}C_{n-r+1}} \cdot b^{(m-r+1)(n-r+1)}.\text{*} \qquad \text{(B)}$$

Proof: By changing the form of two of the factorial numbers in the value of $A\Delta'$, and introducing the factor $[(n-r+1)!]^{(m-r+1)}$ in the numerator and the denominator

$$A\Delta' = \frac{\prod\limits_{i=0}^{i=m-r}(n+i)!}{\prod\limits_{i=0}^{i=m-r}(r-1+i)!(n-r+1)!} \cdot \frac{\prod\limits_{i=0}^{i=m-r} i!(n-r+1)!}{\prod\limits_{i=0}^{i=m-r}(n-r+1+i)!} = \frac{\prod\limits_{i=0}^{i=m-r} {}_{n+i}C_{n-r+1}}{\prod\limits_{i=0}^{i=m-r} {}_{n-r+1+i}C_{n-r+1}}.$$

The corollary follows.

* See paper by Segre, already referred to.

On the Lie-Riemann-Helmholtz-Hilbert Problem of the Foundations of Geometry.

BY ROBERT L. MOORE.

§ 1. *Introduction.*

Concerning Hilbert's paper, "Über die Grundlagen der Geometrie,"* Poincaré says, according to Halsted's translation,† "As regards the ideas of Lie, the progress made is considerable. Lie supposed his groups defined by analytic equations. Hilbert's hypotheses are far more general. Without doubt this is still not entirely satisfactory, since though the *form* of the group is supposed any whatever, its *matter*, that is to say the plane which undergoes the transformations, is still subjected to being a *number-manifold* in Lie's sense. Nevertheless, this is a step in advance, and besides Hilbert analyzes better than anyone before him the idea of *number-manifold* and gives outlines which may become the germ of an assumptional theory of analysis situs."

The present paper contains a set of assumptions Σ in terms of the notions *point*, *region*, and *motion*. Here the space which undergoes the transformations (motions) is *not subjected in advance* to the condition of being a number plane nor is it *presupposed* that the *regions* are in one-to-one correspondence with portions of such a plane. Of course Poincaré's statement that "the form of the group is supposed any whatever" applies to its *presupposed* form. Hilbert's axioms so condition the form of the group in question as to necessitate that it should be simply isomorphic with the group of rigid motions in a space of two dimensions. It is largely, or entirely, a question of analysis. It may be said that Hilbert *analyzes* the group of transformations (motions) but leaves largely unanalyzed the space that undergoes the transformations. In the present treatment the "form" of the transformations and their "matter" (the space that is transformed by them) are subjected to what might be termed a *simultaneous analysis.*

* *Matematische Annalen*, Vol. LVI (1902–03), pp. 381–422.

† "The Bolyai Prize," *Science*, May 19, 1911, p. 765.

§ 2. *Preliminary Explanations and Definitions.*

I consider a class $\overline{S}$ of undefined elements called *points*, an undefined class of sub-classes of $\overline{S}$ called *regions* and an undefined class of one-to-one transformations of $\overline{S}$ into itself called *motions*.* If P is a point of $\overline{S}$, and M is a motion, the point into which P is transformed by M will be denoted by the symbol $M(P)$. If K is a point-set and M is a motion, $M(K)$ will denote the set of all points $M(P)$ for all points P of K.

DEFINITIONS. A point P is said to be a limit point of a point-set K if and only if every region that contains P contains at least one point of K distinct from P. The boundary of a point-set K is the set of all points $[X]$ such that every region that contains X contains at least one point of K and at least one point that does not belong to K. If K is a point-set, K' denotes the set of points composed of K plus its boundary. If R is a region the point-set $\overline{S}-R'$ is called the *exterior* of R. A point in the exterior of R is said to be *without* R.

A set of points is said to be *connected* if however it be divided into two mutually exclusive subsets, one of them contains a limit point of the other one. A set of points is said to be *closed* if it contains all its limit points. A set of points is said to be *continuous* if it is both closed and connected.

A *domain* is a set of points K such that if P is a point of K then there exists a region that contains P and is contained in K.

A set of regions G is said to *cover* a point-set K if each point of K belongs to at least one region of the set G. If for every infinite set of regions G that covers the point-set K there exists a finite subset of G that also covers K, then K is said to *possess the Heine-Borel property.*

A set of points K is said to be *bounded* if there exists a region R such that K is a subset of R'.

If A and B are two distinct points, a *simple continuous arc* from A to B is a continuous bounded point-set that contains A and B, but is disconnected† by the omission of any one of its points other than A and B.

A *simple closed curve* is a continuous bounded point-set which is disconnected by the omission of any two of its points.

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* By a one-to-one transformation of $\overline{S}$ into itself is meant a transformation T such that (1) for each point P of $\overline{S}$ there exists one and only one point $\overline{P}$ of $\overline{S}$ such that T transforms P into $\overline{P}$, (2) for each point $\overline{P}$ of $\overline{S}$ there is one and only one point P of $\overline{S}$ such that T transforms P into $\overline{P}$. *Point* is wholly undefined. *Region* is undefined except in so far as it is understood that every region is *some* sort of collection of points. *Motion* is undefined except in so far as it is understood that every motion is *some* sort of one-to-one transformation of $\overline{S}$ into itself. In addition to this information, no further information (aside from that furnished by the axioms of the system Σ) is presupposed concerning the terms *point, region,* and *motion.*

† A connected point-set K is said to be disconnected by the omission of a proper subset N if $K-N$ is not connected.

§ 3. *The Axioms of* Σ.

AXIOM 1. *There exists at least one region.*

AXIOM 2. *If R and K are regions and R' is a subset of K' then R is a subset of K.*

AXIOM 3. *If the region R_1 contains the point O in common with the region R_2, there exists a region R containing O such that R' is common to R_1 and R_2.*

AXIOM 4. *If R_1 and R_2 are regions and R_2' is a subset of R_1 then $R_1 - R_2'$ is a non-vacuous connected point-set.*

AXIOM 5. *If R_1 and R_2 are regions there exists a region R that contains both R_1' and R_2'.*

AXIOM 6. *Every simple closed curve is the boundary of a region.*

AXIOM 7. *If O is a point and L and N are closed bounded point-sets with no point in common, there exists a region K containing O such that if P is a point in K then every region that contains both a point of L and a point of N can be transformed, by a motion that carries some point of L into O, into a point-set that contains both O and P.*

AXIOM 8. *If R is a region and M is a motion then $M(R)$ is a region.*

AXIOM 9.* *If A, B, C, A', B', C' are points, distinct or otherwise, such that every three regions that contain A, B and C, respectively, can be transformed by some motion into regions containing A', B', and C', respectively, then there exists a motion that transforms A into A', B into B', and C into C'.*

AXIOM 10.† *If M is a motion there exists a motion M^{-1} such that if $M(A) = B$ then $M^{-1}(B) = A$.*

AXIOM 11.‡ *If M and N are motions there exists a motion MN such that, for every point P, $M(N(P)) = MN(P)$.*

AXIOM 12.§ *If R_1 and R_2 are regions bounded respectively by the simple closed curves J_1 and J_2, R_1' and R_2' have no point in common, A_1, B_1, and C_1 are three distinct points on J_1, and A_2, B_2, and C_2 are three distinct points on J_2, and there exist three simple continuous arcs A_1XA_2, B_1YB_2, and C_1ZC_2 such that no two of these arcs have a point in common and no one of them has any point other than an end-point in common either with R_1' or with R_2' and M is a motion such that R_1' and $M(R_2')$ have no point in common and*

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* Cf. Hilbert's Axiom III, *loc. cit.*, p. 169.

† Cf. Hilbert, *loc. cit.*, p. 167.

‡ Cf. Hilbert's Axiom I, *loc. cit.*, p, 167.

§ Cf. J. R. Kline, "A Definition of Sense on Closed Curves in Non-metrical Plane Analysis Situs," *Annals of Mathematics*, Vol. XIX (1918), pp. 185–200. Axiom 12 corresponds to Hilbert's assumption (*loc. cit.*, p. 167) that motion does not change sense on any simple closed curve.

there exist three arcs $A_1\bar{X}M(A_2)$, $B_1\bar{Y}M(B_2)$, and $C_1\bar{Z}M(C_2)$ from A_1 to $M(A_2)$, from B_1 to $M(B_2)$, and from C_1 to $M(C_2)$, respectively, then there exist three such arcs such that no two of them have a point in common and no one of them has any point other other than end-point in common either with R_1' or with $M(R_2')$.

§ 4. *Consequences of Axioms* 1–4 *and* 7–11.

THEOREM 1. *If the point P is a limit point of the point-set K, and M is a motion, then $M(P)$ is a limit point of $M(K)$.*

Proof. If $M(P)$ were not a limit point of $M(K)$ there would exist a region R containing $M(P)$, but no point of $M(K)$ other than $M(P)$. But in this case $M^{-1}(R)$ would be a region containing P but no point of K other than P, contrary to the hypothesis that P is a limit point of K.

THEOREM 2. *No point of a region is a boundary point of that region.*

THEOREM 3. *Every region contains infinitely many points.*

Theorem 3 can be easily proved with the use of Axioms 3 and 4.

THEOREM 4. *If A and B are distinct points, and C is any point whatever, there exists a region containing C which can not be transformed by a motion into a point-set K such that K' contains both A and B.*

Proof. If no region contains the point C then it is vacuously true that if R_1, R_2, and R_3 are three regions containing C there exists a motion $\bar{M}$ such that $\bar{M}(R_1)$ contains A, and $\bar{M}(R_2)$ and $\bar{M}(R_3)$ contain B. It follows by Axiom 9 that there exists a motion that carries C into both A and B. Thus the supposition that there is no region containing C leads to a contradiction. Suppose that every region containing C can be transformed by a motion into a point-set K such that K' contains both A and B. By Axiom 3, if R is a region containing C, there exists a region $\bar{R}$ containing C such that $\bar{R}'$ is a subset of R. By hypothesis there exists a motion M such that $[M(\bar{R})]'$ contains both A and B. By Theorem 1 $[M(\bar{R})]'=M(\bar{R}')$. Hence $M(R)$ contains both A and B. It follows by Axiom 9 that there exists a motion that transforms C into both A and B. Thus the supposition that Theorem 4 is false leads to a contradiction.

THEOREM 5. *If L and N are two closed, bounded point-sets with no point in common, and O is any point whatever, there exists a region R containing O such that R' can not be transformed by a motion into a point-set that contains both a point of L and a point of N.*

Proof. By Axiom 7 and Theorem 3 there exists a point P distinct from O such that every region that contains both a point of L and a point of N can be transformed by a motion into a point-set that contains both O and P. By Theorem 4 there exists about* O a region $\bar{R}$ which can not be moved into a point-set containing both O and P. If there should exist a motion M such that $M(\bar{R})$ contains both a point of L and a point of N, then there would exist a motion $\bar{M}$ such that $\bar{M}M(\bar{R})$ contains both O and P. But this would involve a contradiction. By Axiom 3 there exists about O a region R such that R' is a subset of $\bar{R}$. The region R clearly satisfies the condition of Theorem 5.

Theorem 6. *If P is a limit point of $M+N$ it is a limit point either of M or of N.*

Theorem 6 can be proved with the use of Axiom 3.

Theorem 7. *If the point P is a limit point of the point-set M, then every region that contains P contains infinitely many points of M.*

Proof. Suppose the region R contains the point P and has in common with M only a finite set of points $P_1, P_2, P_3, \ldots, P_n$ distinct from P. By Theorem 4 for each $i (1 \leqq i \leqq n)$ there exists about P a region R_i that can not be transformed by a motion into a point-set containing P and P_i. But there exists a motion that leaves all points fixed. This motion carries R_i into R_i. But R_i contains P. It follows that R_i does not contain P_i. The regions R_1 and R_2 contain in common a region K_2 containing P. The region K_2 contains neither P_1 nor P_2. Similarly there exists in K_2 a region K_3 that contains P, but no one of the points P_1, P_2, P_3. This process may be continued. It follows that there exists about P a region K_n that lies in R but contains no point of the set $P_1, P_2, P_3, \ldots, P_n$. Hence P is not a limit point of M. Thus the supposition that Theorem 7 is false leads to a contradiction.

Theorem 8. *If the region R contains a point O in the region K and a point P without K, then it contains a point on the boundary of K.*

Proof. By Axiom 3 there exists about O a region R_1 which is a subset both of R and of K. By Theorem 3 there exists in R_1 a point $\bar{P}$ distinct from O. By Theorem 7 and Axiom 3 there exists about O a region R_2 such that R_2' is a subset of $R_1 - \bar{P}$. By Axiom 4 $R - R_2'$ is connected. But it contains the point $\bar{P}$ in K and the point P without K. Hence it contains a point of the boundary of K.

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* In this connection "about" is synonymous with "containing."

THEOREM 9. *If O is a point there exists a countably infinite sequence of regions $R_1, R_2, R_3, \ldots$, such that* (1) *P is the only point they have in common,* (2) *if n is a positive integer and M is a motion such that $M(R_{n+1})$ contains O, then $M(R'_{n+1})$ is a subset of R_n,* (3) *if R is a region containing O there exists an n such that R_n is a subset of R.*

Proof. By Axiom 1 and Theorems 3 and 4 there exists a region K_1 containing O. By Axiom 3 there exists a region K_2 containing O such that K'_2 is a subset of K_1 and a region K_3 containing O such that K'_3 is a subset of K_2. It follows by two applications of Axiom 4 that $K_1 - K'_3$ is a connected point-set containing at least two distinct points. Let $\bar{P}$ denote a definite point of $K_1 - K'_3$. Then $\bar{P}$ is a limit point of $K_1 - K'_3 - \bar{P}$. It follows that there exists a countable sequence of points $\bar{P}_1, \bar{P}_2, \bar{P}_3, \ldots$ all distinct from $\bar{P}$ such that $\bar{P}$ is a limit point of the point-set $\bar{P}_1 + \bar{P}_2 + \bar{P}_3 + \ldots$. By Theorem 7 and Axiom 7 there exists a motion $\bar{M}$ such that $\bar{M}(\bar{P}) = O$. For every n let P_n denote $\bar{M}(\bar{P}_n)$. Then O is a limit point of the point-set $P_1 + P_2 + P_3 + \ldots$. By Theorem 4 there exists about O a region R_1 such that if M is a motion that transforms R_1 into a point-set containing O then $M(R'_1)$ does not contain P_1. By Theorems 6, 7 and 5 there exists a region R_2 containing O such that if M is a motion that transforms R_2 into a point-set containing O, then $M(R'_2)$ contains no point of the closed, bounded point-set $P_2 + L_1$ where L_1 is the boundary of R_1. It follows by Theorem 8 that, for every such motion M, $M(R'_2)$ is a subset of R_1. This process may be continued. It follows that there exists a sequence of regions $R_1, R_2, R_3, \ldots$ containing O such that if n is a positive integer and M is a motion that transforms R_{n+1} into a point-set containing O, then $M(R'_{n+1})$ contains no point of the point-set $P_1 + P_2 + P_3 + \ldots + P_n + S - R_n$. The sequence $R_1, R_2, R_3, \ldots$ satisfies the requirements of Theorem 9. That it satisfies requirement (2) is obvious. Suppose it does not satisfy both (1) and (3). Then there exists a point P distinct from O and a region R containing O such that for every n R_n contains a point of the closed and bounded point-set $P + L$ where L is the boundary of R. It follows by Axiom 7 that there exists a positive integer m such that, for every n, R_n can be transformed by some motion into a point-set containing P_m and O. But R_{m+1} can not be moved into such a point-set. Thus the supposition that $R_1, R_2, R_3, \ldots$ does not satisfy requirements (1) and (3) has led to a contradiction.

THEOREM 10. *Every region is a connected set of points.*

Proof. Suppose R_1 is a region. There exists in R_1 a point P. By Theorem 9 there exists a sequence of regions $R_2, R_3, R_4, \ldots$, all lying in R_1

and having in common only the point P and such that (1) for each n, R'_{n+1} is a subset of R_n, (2) if R is a region containing P there exists an n such that R contains R_n. By Axiom 4, $R_1 - R'_n$ is connected. But

$$R_1 = (R_1 - R'_2) + (R_1 - R'_3) + (R_1 - R'_4) + \ldots . + P,$$

$R_1 - R'_n$ is a subset of $(R_1 - R'_{n+1})$ and P is a limit point of

$$(R_1 - R'_2) + (R_1 - R'_3) + (R_1 - R'_4) + \ldots ..$$

It easily follows that R_1 is connected.

THEOREM 11. *Every boundary point of a region is a limit point of the exterior of that region.*

Proof. Suppose the boundary of the region R contains a point X which is not a limit point of $\bar{S} - R'$. Then there exists a region $\bar{R}$ that contains X and lies wholly in R'. It follows that $\bar{R}'$ is a subset of R'. Therefore, by Axiom 2, $\bar{R}$ is a subset of R. Thus X belongs to R and is therefore not a boundary point of R.

THEOREM 12. *If $R_1, R_2, R_3, \ldots .$ is a sequence of regions closing down* on the point O, $M_1, M_2, M_3, \ldots .$ are motions and L and N are two closed and bounded point-sets with no point in common, then there do not exist infinitely many positive integers n such that $M_n(R'_n)$ contains both a point of L and a point of N.*

Theorem 12 is a consequence of Theorem 5.

THEOREM 13. *If $R_1, R_2, R_3, \ldots .$ is a sequence of regions closing down on the point O, $M_1, M_2, M_3, \ldots .$ are motions, and $A_1, A_2, A_3, \ldots .$ and $B_1, B_2, B_3, \ldots .$ are two infinite sequences of points such that, for every n, A_n and B_n are both in $M_n(R_n)$, then if $A_1 + A_2 + A_3 + \ldots .$ has a limit point every such point is also a limit point of $B_1 + B_2 + B_3 + \ldots ..$*

Proof. Suppose X is a limit point of $A_1 + A_2 + A_3 + \ldots .$, and R is a region containing X. By Axiom 3 there exists a region $\bar{R}$ containing X such that $\bar{R}'$ is a subset of R. The region $\bar{R}$ contains infinitely many distinct points $A_{n_1}, A_{n_2}, A_{n_3}, \ldots .$ of the sequence $A_1, A_2, A_3, \ldots ..$ It follows with the help of Theorems 8 and 12 that, for infinitely many positive integers i, R_{n_i}, and therefore B_{n_i}, is a subset of R. It follows that X is a limit point of $B_1 + B_2 + B_3 + \ldots ..$

THEOREM 14. *Every bounded infinite set of points has at least one limit point.*

*A sequence of regions $R_1, R_2, R_3, \ldots .$ is said to *close down on* the point O if it satisfies with respect to O all the requirements of Theorem 9.

Proof. Suppose that R is a region and that $X_1, X_2, X_3, \ldots.$ is an infinite set of distinct points lying in R' and having no limit point. Let $R_1, R_2, R_3, \ldots.$ denote a sequence of regions closing down on a point O. For each n there exists a motion M_n such that $M_n(O) = X_n$. For each n there exists in the region $M_n(R_n)$ a point $\bar{X}_n$ distinct from every point of the set $X_1, X_2, X_3, \ldots.$ and lying in R'. By Theorem 13 the point-set $\bar{X}_1 + \bar{X}_2 + \bar{X}_3 + \ldots.$ has no limit point. Thus $X_1 + X_2 + X_3 + \ldots.$ and $\bar{X}_1 + \bar{X}_2 + \bar{X}_3 + \ldots.$ are closed, bounded point-sets with no point in common. But, for every n, $M_n(R_n)$ contains a point of each of these sets. This is contrary to Theorem 12.

THEOREM 15. *There does not exist a compact* point-set K and an uncountably infinite set G of mutually exclusive regions such that every region of the set G contains a point of K.*

Proof. Suppose there does exist a compact point-set K and an uncountable set of regions G satisfying such conditions. Let $R_1, R_2, R_3, \ldots.$ denote a sequence of regions closing down on some point O. If X is a point of K lying in a region g of the set G then for each n there exists a motion M_n such that $M_n(O) = X$. By Theorems 8 and 12 there exists a positive integer m such that $M_m(R_m)$ is a subset of g. It follows by Zermelo's Postulate that there exists a set $\bar{G}$ of regions such that (1) each region of the set G contains one and only one region of the set $\bar{G}$, (2) for each region $\bar{g}$ of the set $\bar{G}$ there exists a motion that transforms some region of the set $R_1, R_2, R_3, \ldots.$ into $\bar{g}$ and transforms the point O into a point of K that lies in $\bar{g}$. In view of the fact that G is an uncountable set it follows that there exists a positive integer n and a set of motions $M_1, M_2, M_3, \ldots.$ such that, for every j, $M_j(O)$ is a point of K and such that no two of the regions $M_1(R_n), M_2(R_n), M_3(R_n), \ldots.$ have a point in common. By hypothesis the set of points $M_1(O) + M_2(O) + M_3(O) \ldots.$ has at least one limit point Z. There exists a motion $\bar{M}$ such that $\bar{M}(R_{n+1})$ contains Z. There exists k such that $M_k(O)$ is in $\bar{M}(R_{n+1})$. Hence O is in $M_k^{-1}\bar{M}(R_{n+1})$. It follows that $M_k^{-1}\bar{M}(R'_{n+1})$ is a subset of R_n. Hence $\bar{M}(R'_{n+1})$ is a subset of $M_k(R_n)$. Therefore $M_k(R_n)$ contains Z. Hence there exists an index i distinct from k such that $M_k(R_n)$ contains $M_i(O)$. It follows that $M_k(R_n)$ and $M_i(R_n)$ have a point in common. Thus the supposition that Theorem 15 is false leads to a contradiction.

THEOREM 16. *Every compact set of points is a subset of a compact domain.*

* A set of points K is said to be *compact* if every infinite subset of K has at least one limit point. Cf. M. Fréchet, "Sur quelques points du calcul fonctionnel," *Rendiconti del circolo matematico di Palermo*, Vol. XXII (1906), p. 6.

Proof. Let $R_1, R_2, R_3, \ldots$ be a sequence of regions closing down on a point O. There exists an index m greater than 1 such that if M is a motion such that $M(R_m)$ contains a point of R_m then $M(R_m)$ is a subset of R_1. Suppose there exists a compact set of points K which is not a subset of a compact domain. Then there must exist an infinite set of distinct points $P_1, P_2, P_3, \ldots$ lying in K, an infinite set of distinct points $\overline{P}_1, \overline{P}_2, \overline{P}_3, \ldots$ such that $\overline{P}_1+\overline{P}_2+\overline{P}_3+\ldots$ has no limit point and a set of motions $M_1, M_2, M_3, \ldots$ such that for every n the region $M_n(R_m)$ contains both P_n and $\overline{P}_n$. The point-set $P_1+P_2+P_3$ has at least one limit point $\overline{O}$. There exists a motion $\overline{M}$ such that $\overline{M}(O)=\overline{O}$. There exists an infinite set of distinct positive integers n_1, $n_2, n_3, \ldots$ such that the points $P_{n_1}, P_{n_2}, P_{n_3}, \ldots$ are all in $\overline{M}(R_m)$. Each of the regions $M_{n_1}(R_m), M_{n_2}(R_m), M_{n_3}(R_m), \ldots$ is a subset of $\overline{M}(R_1)$. Hence the infinite point-set $\overline{P}_{n_1}+\overline{P}_{n_2}+\overline{P}_{n_3}+\ldots$ is a subset of $\overline{M}(R_1)$. It follows by Theorem 14 that $\overline{P}_1+\overline{P}_2+\overline{P}_3+\ldots$ has a limit point. Thus the supposition that Theorem 16 is false has led to a contradiction.

THEOREM 17. *If the compact point-set T is covered by an uncountably infinite set of regions G, then T is covered by some countable subset of G.*

Proof. By Theorem 16 the compact point-set T is a subset of some compact domain K. Let $R_1, R_2, R_3, \ldots$ denote a sequence of regions closing down on a point O. For each point P of T there exists an integer n_P greater than 1, and a motion M_P such that the region $M_P(R_{n_P-1})$ contains P and lies both in K and in some region of the set G. Let $\overline{G}$ be the set of all regions $M_P(R_{n_P})$ for all points P of T. There exists a finite or countably infinite set of distinct positive integers $m_1, m_2, m_3, \ldots$ and a sequence $\overline{G}_{m_1}, \overline{G}_{m_2}, \overline{G}_{m_3}, \ldots$ of subsets of $\overline{G}$ such that (1) $\overline{G}=\overline{G}_{m_1}+\overline{G}_{m_2}+\overline{G}_{m_3}+\ldots$ (2) for every i, $\overline{G}_{m_i}$ is the set of all regions $M_P(R_{n_P})$ for which $n_P=m_i$. For every i let T_i denote the set of all those points of T which are covered by the set of regions $\overline{G}_{m_i}$. The regions of the set $\overline{G}_{m_i}$ can be arranged in a well-ordered sequence $g_{i1}, \bar{g}_{i2}$, $\bar{g}_{i3}, \ldots \bar{g}_{i\omega}, \ldots$. Call this well-ordered sequence β. Let g_{i2} be the first region in the sequence β that contains a region that has no point in common with g_{i1}. Let g_{i3} be the first one following g_{i1} that contains a region that has no point in common with g_{i1} or with g_{i2}. Continue this process thus obtaining a well-ordered sequence γ such that (1) every element of γ is an element of β, (2) if γ_1 is any subsequence of γ such that there exists an element of γ that follows all the elements of γ_1, then the first element of γ that follows all the elements of γ_1 is the first element of β that contains a region that has no point in common with any of the elements of γ_1. There is not an uncountable

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infinity of elements in the sequence γ. For if there were, there would exist in K an uncountable infinity of distinct regions, no two of which have a point in common, which is contrary to Theorem 15. It follows that there exists a countable subset G_i of the set of regions $\overline{G}_{m_i}$ such that every point of T_i either is in a region of the set G_i or is a limit point of the point-set L_i obtained by adding together all the regions $M_{P_1}(R_{m_i})$, $M_{P_2}(R_{m_i})$, $M_{P_3}(R_{m_i})$, ... of the set G_i. Let $\overline{\overline{G}}_i$ denote the set of regions $M_{P_1}(R_{m_i-1})$, $M_{P_2}(R_{m_i-1})$, $M_{P_3}(R_{m_i-1})$, ... The set $\overline{\overline{G}}_i$ covers T_i. For suppose there exists a point X of T_i which lies in no region of the set $\overline{\overline{G}}_i$. Then X is a limit point of L_i. There exists a positive integer k_i such that the region R_{k_i} can not be transformed by a motion into a point-set containing both a point in R_{m_i} and a point in $\overline{S}-R_{m_i-1}$. But there exists a motion $\overline{M}$ such that $\overline{M}(R_{k_i})$ contains X. The region $\overline{M}(R_{k_i})$ contains a point of L_i and therefore, for some j, a point in common with $M_{P_j}(R_{m_i})$. If it also contained a point in common with $\overline{S}-M_{P_j}(R_{m_i-1})$ then $M_{P_j}^{-1}\overline{M}(R_{k_i})$ would contain a point of R_{m_i} and a point of $\overline{S}-R_{m_i-1}$. But this is impossible. It follows that, for every i, T_i is covered by $\overline{\overline{G}}_i$. But every region of $\overline{\overline{G}}_i$ is a subset of some region of G, and T is the sum of the finite or countably infinite set of point-sets T_1, T_2, T_3, ... It follows that T is covered by a countable subset of G.

THEOREM 18. *If a closed and compact point-set is covered by an infinite set of regions then it is also covered by some finite subset of that set of regions.*

Proof. By Theorem 17 if a compact point-set is covered by an infinite set of regions G it is covered by a countable subset of G. But if a closed and compact point-set is covered by a *countable* set of regions then * it is covered by a finite subset of that countable set.

THEOREM 19. *If H' is a closed and bounded set of points there exists an infinite sequence of regions $K_{H'1}$, $K_{H'2}$, $K_{H'3}$, ... such that* (1) *if m is a positive integer and P is a point of H', there exists an integer n, greater than m, such that $K_{H'n}$ contains P,* (2) *if P and $\overline{P}$ are distinct points of H' lying in a region R, there exists an integer δ such that if $n>\delta$ and $K_{H'n}$ contains P then $K'_{H'n}$ is a subset of $R-\overline{P}$.*

Proof. Let R_1, R_2, R_3, ... be a set of regions closing down on a point O. If X is a point of H' and n is a positive integer, the region R_n can be transformed by a motion into a region K_{Xn} containing X. For any fixed n consider

* Cf. F. Hausdorff, "Grundzüge der Mengenlehre," Veit & Co., Leipzig, 1914, p. 231.

the set of all such K_{X_n}'s for all points X of H'. By Theorems 14 and 18 there exists a finite number of these K_{X_n}'s, say $K_n^{(1)}, K_n^{(2)}, K_n^{(3)}, \ldots, K_n^{(m_n)}$ covering H'. Let n take on successively all positive integral values. Let $K_{H'1}$, $K_{H'2}$, $K_{H'3}, \ldots, K_{H'm_1}$ denote $K_1^{(1)}, K_1^{(2)}, K_1^{(3)}, \ldots, K_1^{(m_1)}$, respectively. For $n \geqq 1$ let $K_{H', m_n+1}, K_{H', m_n+2}, K_{H', m_n+3}, \ldots, K_{H'm_{n+1}}$ denote $K_{n+1}^{(1)}, K_{n+1}^{(2)}, K_{n+1}^{(3)}, \ldots, K_{n+1}^{m_{n+1}}$, respectively. The infinite sequence of regions $K_{H'1}, K_{H'2}, K_{H'3}, \ldots$ so obtained satisfies all the requirements of Theorem 19.

Theorem 20. *If H is a region there exists an infinite sequence of regions $K_{H_1}, K_{H_2}, K_{H_3}, \ldots$ such that each K_{H_n} is a subset of H, and such that if, in the statement of Conditions* (1) *and* (2) *of Theorem* 19, *H' is replaced by H, the resulting conditions are fulfilled.*

Proof. Let $K_{H'1}, K_{H'2}, K_{H'3}, \ldots$ be a sequence of regions satisfying all the requirements of Theorem 19. Let K_{H1} denote the first region of this sequence that lies in H. Let K_{H2} denote the first region that follows K_{H1} in the same sequence and lies in H. If this process is continued there will be obtained a sequence of regions satisfying the requirements of Theorem 20.

§ 5. *Consequences of Axioms* 1–12.

Theorem 21. *If R is a region $\overline{S}-R'$ is a connected set of points.*

Proof. That $\overline{S}-R'$ exists and contains infinitely many points may be easily proved with the use of Axioms 5 and 4. Suppose A and B are two points in $\overline{S}-R'$. There exist regions R_A and R_B containing A and B, respectively. By two applications of Axiom 5 there exists a region K that contains R', R'_A and R'_B, and therefore contains A and B. By Axiom 4, $K-R'$ is connected. Thus every two points of $S-R'$ lie together in a connected subset of $S-R'$. It easily follows that $S-R'$ is connected.

Theorem 22. *There exists an infinite set of points that has no limit point.*

Proof. Suppose on the contrary that every infinite set of points has at least one limit point. Then $\overline{S}$ is compact and closed. But every point of $\overline{S}$ is in some region. Hence by Theorem 18 there exists a finite set of regions R_1, $R_2, R_3, \ldots, R_n$, such that every point is in some region of this set. But by n applications of Axiom 5 there exists a region R that contains all the regions $R_1, R_2, R_3, \ldots, R_n$. Hence R contains all points. But, by Axiom 5, there exists a region K that contains R' and, by Axiom 4, K contains at least one point that is not in R. Thus the supposition that Theorem 22 is false has led to a contradiction.

With the use in particular of Axioms 3 and 5, and Theorems 14, 10 and 20, the truth of the following theorem may be established by methods in large part similar to or identical with those employed in the proof of Theorem 15 on pages 136–139 of my paper "On the Foundations of Plane Analysis Situs."*

THEOREM 23. *Every two points of a connected domain are the extremities of a simple continuous arc that lies wholly in that domain.*

THEOREM 24. *If O is a point in a region R there exists a simple closed curve that lies in R and encloses O.*

Proof. There exists a region $\bar{R}$ which can not be transformed by a motion into a region L such that L' contains both O and a point of $S-R$. By Theorems 23 and 10, if A and B are two points in the region $\bar{R}$, there exists an arc AXB lying wholly in $\bar{R}$. There exists a region K containing X such that K' is a subset of $\bar{R}-A-B$. There exists an arc AYB that lies wholly in the domain $\bar{R}-K'$. It is clear † that the point-set composed of the arcs AXB and AYB together contains at least one simple closed curve J. By Axioms 6 and 2, and Theorem 21, I, the interior of J, is a subset of $\bar{R}$. There exists a motion M such that $M(I)$ contains O. The closed curve $M(J)$ lies in R and encloses O.

THEOREM 25. *If J and C are simple closed curves, O is a point on J but not on C, A_1 and A_2 are distinct points common to C and J, and A_1XA_2 is an arc on C such that $\underparen{A_1XA_2}$ ‡ lies within J, then there exist two points O_1 and O_2 distinct from O such that*

(1) *O_1 and O_2 lie on the intervals A_1O and A_2O of the arc A_1OA_2 of the closed curve J.*

(2) *There is on the curve C an arc O_1YO_2 such that $\underparen{O_1YO_2}$ is within J.*

(3) *If B_1 and B_2 are points on the intervals O_1O and O_2O, respectively, of the arc A_1OA_2 of J, such that there exists on C from B_1 to B_2 an arc which, except for its end-points, lies entirely within J, then $B_1=O_1$ and $B_2=O_2$.*

For a proof of Theorem 25 see pages 152 and 153 of F. A.

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* *Transactions of the American Mathematical Society*, Vol. XVII (1916), pp. 131–164. Hereafter this paper will be referred to as F. A.

† For a proof that simple continuous arcs and simple closed curves as defined in the present paper have certain fundamental properties including properties of linear and cyclical order respectively, see my paper, "Concerning Simple Continuous Curves." See also F. A., pp. 139 and 140.

‡ If ABC is an arc, $\underparen{ABC}$ denotes the point-set $ABC-A-B$. Likewise if AB is an arc, $\underparen{AB}$ denotes the point-set $AB-A-B$.

THEOREM 26. *If P is a point on a closed curve J, and $\bar{J}$ is a closed curve enclosing P and containing at least one point within J, then there exist two closed curves Q and $\bar{Q}$ such that* (1) *every point of Q belongs either to J or to $\bar{J}$ and so does every point of $\bar{Q}$,* (2) *the curves Q and $\bar{Q}$ contain in common a segment of J that contains P,* (3) *the interiors of Q and $\bar{Q}$ are both subsets of the interior of $\bar{J}$,* (4) *every point within Q is within J and every point within $\bar{Q}$ is without J.*

Proof. If the curve J had no point without $\bar{J}$, the interior of J would be a subset of the interior of $\bar{J}$ and therefore could not contain a point of $\bar{J}$. It follows that J contains a point C without $\bar{J}$. The curve J is the sum of two simple continuous arcs PEC and PFC. Let A and B denote the first points that the arcs PEC and PFC, respectively, have in common with $\bar{J}$. The curve $\bar{J}$ is the sum of two arcs $A\bar{E}B$ and $A\bar{F}B$ (Fig. 1). Let H denote the interior of

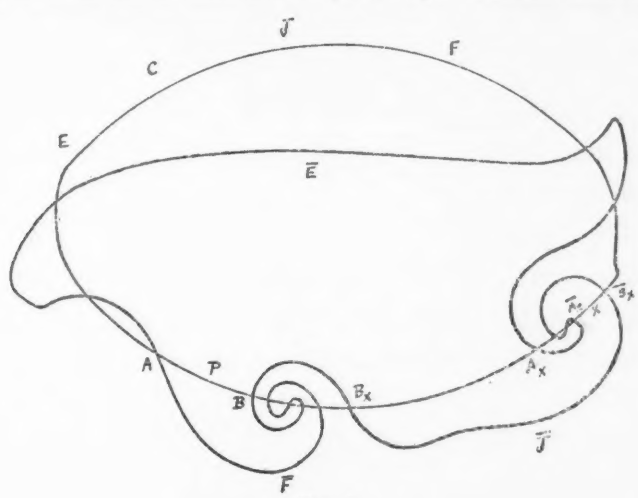

FIG. 1.

the closed curve $A\bar{E}BPA$ formed by the arc $A\bar{E}B$ and the interval APB of the curve J, and let K denote the interior of the closed curve $A\bar{F}BPA$ formed by the arc $A\bar{F}B$ and the same interval APB. By Theorems 24 and 25 of F. A. the point-sets H, K and APB are mutually exclusive and $H+K+\underset{\smile}{APB}=\bar{I}$, the interior of $\bar{J}$. If the curve J contains a point X in H, then J has in common with the curve $A\bar{E}BPA$ two distinct points $\bar{A}_x$ and $\bar{B}_x$ such that the segment $\bar{A}_xX\bar{B}_x$ of J is a subset of H. By Theorem 25 there exists on J an arc A_xB_x such that (1) A_x is on the subinterval $\bar{A}_xP$ of the interval $\bar{B}_x\bar{A}_xP$ of the curve $A\bar{E}BPA$, (2) B_x is on the subinterval $\bar{B}_xP$ of the interval $\bar{A}_x\bar{B}_xP$ of $A\bar{E}BPA$, (3) $\underset{\smile}{A_xB_x}$ is in H, (4), if Y is a point on the subinterval A_xP of the interval

$\overline{B}_x\overline{A}_xP$ of the curve $A\overline{E}BPA$, and Z is a point on the subinterval B_xP of the interval $\overline{A}_x\overline{B}_xP$ of $A\overline{E}BPA$ such that the interval ACB of J contains a segment that has Y and Z as end-points and lies entirely in H, then $Y=A_x$ and $Z=B_x$. It is clear that the arc A_xB_x is completely determined by the point X. For every X on ACB that lies in H construct the corresponding A_xB_x. Let $\overline{h}$ denote the point-set which is composed of all the A_xB_x's so constructed together with every point F on $A\overline{E}BPA$ which has the property that for no X is F separated from P by A_x and B_x. It may be easily proved that $\overline{h}$ is a simple closed curve. That $\overline{H}$, the interior of $\overline{h}$, is a subset of H, and therefore of the interior of $\overline{J}$ is a consequence of Axioms 6 and 2 and Theorem 21. It is clear that $\overline{h}$ contains APB and that $\overline{H}$ contains no point of J. Similarly there exists a closed curve $\overline{k}$ containing APB such that every point of $\overline{k}$ belongs either to J or to $A\overline{F}BPA$, and such that (1) $\overline{K}$, the interior of $\overline{k}$, is a subset of K, (2) $\overline{K}$ contains no point of J. The point-set $\overline{h}+\overline{k}-\underbrace{ABP}$ is evidently a simple closed curve α. If $\underbrace{APB}$ were in the exterior of α, then by Theorem 27 of F. A., either $\overline{H}$ would be a subset of $\overline{K}$, or $\overline{K}$ would be a subset of $\overline{H}$, neither of which is possible in view of the facts that $\overline{H}$ and $\overline{K}$ are subsets of H and K respectively, and H and K have no point in common. It follows that APB is within α. Hence, by Theorem 25 of F. A., R, the interior of α, $=APB+\overline{H}+\overline{K}$. Since R contains P, a boundary point of J, it must contain a point P_1 within J and a point P_2 without J. Let Q denote that one of the curves $\overline{h}$ and $\overline{k}$ that encloses P_1 and let $\overline{Q}$ denote the other one. Since the interior of Q contains a point within J and no point on J it must lie entirely within J. Hence P_2 is within $\overline{Q}$, and $\overline{Q}$ is entirely without J. The curves Q and $\overline{Q}$ fulfill all the requirements of Theorem 26.

THEOREM 27. *If R and $\overline{R}$ are Jordan regions,* and P is a point in $\overline{R}$ and on the boundary of R, there exist in $\overline{R}$ two Jordan regions K and $\overline{K}$ such that $\overline{K}$ contains P, K lies in R, and all those points of the boundary of R that lie in $\overline{K}$ are points also of the boundary of K.*

THEOREM 28. *If R and $\overline{R}$ are Jordan regions and P is a point in $\overline{R}$ and on the boundary of R, there exist in $\overline{R}$ two regions L and $\overline{L}$ such that $\overline{L}$ contains P, L lies in $S-R'$ and all those points of the boundary of R that lie in $\overline{L}$ are points also of the boundary of L.*

Proofs of Theorems 27 *and* 28. Let J and $\overline{J}$ denote the boundaries of R and $\overline{R}$ respectively, and let K and L respectively denote the interiors of the

* A *Jordan* region is the interior of a simple closed curve.

curves Q and $\bar{Q}$ that satisfy with respect to J, $\bar{J}$ and P the requirements of Theorem 26. Let $\bar{K}$ and $\bar{L}$ denote two Jordan regions both of which contain P and lie in $\bar{R}$, but neither of which contains any point of J that is not on the interval APB described in the above proof of Theorem 26. It is clear that the regions K and $\bar{K}$ satisfy the requirements of Theorem 27 and that L and $\bar{L}$ satisfy those of Theorem 28.

If Theorems 20, 10, 21, 18 and 14, 22, 26, 27 and Axiom 6 of the present paper are compared with Axioms 1–8 of F. A., it will be seen that if the term *region* employed in the latter axioms is restricted to mean Jordan region, the so modified axioms hold true* for the set of all points S lying in a given region R of our space $\bar{S}$. Furthermore, in view of Theorem 24 it is clear that if a point is a limit point of a point-set in accordance with the definition given in the present paper it is also a limit point of that point-set in accordance with the definition obtained by substituting "Jordan region" for "region" in that definition and conversely. In view of a theorem established in my paper "Concerning a Set of Postulates for Plane Analysis Situs,"† it follows that the following theorem holds true.

THEOREM 29. *If R is a region, then between the points of R and the set of all sensed pairs of real numbers there is a one-to-one correspondence which is continuous in the sense that in R the point P is a sequential limit point*‡ *of the sequence of points $P_1, P_2, P_3, \ldots$ if and only if $\lim_{n=\infty} x_n = x$ and $\lim_{n=\infty} y_n = y$ where (x_n, y_n) and (x, y) are the number pairs that correspond to P_n and to P respectively.*

Definitions. If A is a point, a rotation about A is a motion M such that $M(A)=A$. If A and B are two distinct points the set of all points $[X]$ such that B can be transformed into X by a rotation about A is a *circle*. The point A is the *center* of this circle. The notation k_{AB} will be used to denote the circle which contains B and has its center at A. A circle will be called *proper* if it is a simple closed curve and its center lies within it. A proper circle k_{OA} is said to have a *proper interior* if for every point P, within k_{OA} and distinct from O, the circle k_{OP} is proper.

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* Of course here Axiom 5 would be interpreted to mean that there exists in the region R an infinite set of points that has no limit point in R, and Axiom 2 would be interpreted as meaning that if J is a simple closed curve lying in R the interior of J is connected.

† *Transactions of the American Mathematical Society*, Vol. XX (1919), pp. 169–178.

‡ The point P is said to be a *sequential limit point* of the infinite sequence of points $P_1, P_2, P_3, \ldots$ if and only if for every region K containing P there exists an integer δ such that if $n > \delta$ then P_n lies in K.

THEOREM 30.* *If O is a point there exists a simple closed curve J enclosing O such that if P is a point on or within J the circle k_{OP} contains infinitely many points.*

Proof. There exists a point A distinct from O and a region R_A containing A such that R'_A does not contain O. By Axioms 7 and 3, and Theorems 5 and 7 there exists a region R containing O such that if P is a point in R then every region that contains both A and O can be transformed by some motion into a region containing P, but such that R can not be transformed by a rotation about O into a point-set containing a point of R'_A. There exists a simple closed curve J that encloses O and lies wholly in R. Let P denote a definite point within or on J but distinct from O. There exists about O, and within J, a region K such that K' can not be transformed by a rotation about O into a point-set that contains P. It may bè easily established with the aid of Theorem 29 that there exists an infinite set $g_1, g_2, g_3, \ldots.$ of distinct regions each containing O and A such that if n_1 and n_2 are two distinct positive integers, every point that is common to g_{n_1} and g_{n_2} is either in R_A or in K. For each n there exists a motion M_n and a point P_n in g_n such that $M_n(P_n)=P$. For each n, P_n is in the exterior both of R_A and of K. Hence the points P_1, $P_2, P_3, \ldots.$ are all distinct. But by Axiom 10 these points are all on the circle k_{OP}.

With the help of Theorems 29 and 30 the next two theorems can be proved by methods wholly or in large part identical with those employed by Hilbert † in his proofs of more or less closely related theorems.

THEOREM 31. *If O is a point in a region R and there exists a region $\bar{R}$ such that every rotation about O throws R into a subset of $\bar{R}$, and every circle with center at O that contains a point of R' contains infinitely many points, then if P is a point of R' distinct from O the circle k_{OP} is a proper circle with proper interior.*

THEOREM 32. *If k is a proper circle with center at O and with a proper interior then* (1) *every rotation about O that leaves fixed one point within k leaves fixed every point within k,* (2) *if X is a point within k, and M_1 and M_2 are rotations about O such that $M_1(X)=M_2(X)$ then, for every point P within or on k, $M_1(P)=M_2(P)$.*

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* Cf. Hilbert's Axiom II, *loc. cit.*, p. 168.

† Cf. in particular pp. 173–191, *loc. cit.* On page 179, in lines 5 and 6, Hilbert says "so ist W* offenbar ein Weg, welche K_3 und K_4 innerhalb dieser neuen Kurve verbindet." It is possible that W* should lie *without* this "neuen Kurve." This possibility does not, however, in any way invalidate what follows.

THEOREM 33. *If $\bar{k}$ is a proper circle with a proper interior there exists a proper circle k^* with a proper interior such that k^* and $\bar{k}$ are concentric and $\bar{k}$ is within k^*.*

Proof. Let O denote the center of $\bar{k}$. There exists a region R such that $\bar{k}$ and its interior are subsets of R. If P is a point on $\bar{k}$ there exists about P a region $\bar{R}$ which does not contain O and which can not be transformed by any motion into a region containing both a point of $\bar{k}$ and a point on the boundary of R. There exists about P a region $\bar{K}$ such that if X and Z are two distinct points in $\bar{K}$, the circle k_{XZ} is a simple closed curve enclosing X. The regions $\bar{R}$ and $\bar{K}$ contain in common a region K that contains P. There exist two points A and B on $\bar{k}$ and an arc AXB, lying, except for its end-points, entirely without $\bar{k}$, such that the closed curve α formed by the arc AXB and the arc APB of the circle $\bar{k}$ lies entirely within K.

Suppose that, for some point W within the closed curve α, the circle k_{OW} contains only a finite number of points. Then there must exist a motion $\bar{M}$ such that $\bar{M}(O)=O$, and such that the circles $\bar{M}(k_{WP})$ and $\bar{k}$ have in common an uncountable set of points N such that if X and Y are any two points of N, there exists a motion M such that $M(O)=O$, $M(X)=Y$ and $M(\bar{W})=\bar{W}$ where $\bar{W}$ is the center of $\bar{M}(k_{WP})$. It follows that there exists in the set N two points C and D such that if M_0 is a motion, such that $M_0(C)=D$, $M_0(O)=O$, and $M_0(\bar{W})=\bar{W}$, then there exists no positive integer n such that $M_0^n(C)=C$. It follows that if L denotes the set of all points $[\bar{P}]$ such that, for some positive integer m, $M_0^m(C)=\bar{P}$ then L is everywhere dense both on $\bar{k}$ and on $\bar{M}(k_{WP})$. It follows that $\bar{M}(k_{WP})$ is identical with $\bar{k}$. But this is impossible in view of the fact that $\bar{M}(k_{WP})$ encloses the point $\bar{W}$ which lies without $\bar{k}$.

It follows that for every point W within α the circle k_{OW} contains infinitely many points. Let β denote the simple closed curve composed of the arc AXB of the curve α together with that arc of $\bar{k}$ which has A and B as end-points but does not contain P. The interior of β equals the interior of $\bar{k}$ plus the interior of α plus the segment APB of α. It follows with the aid of Theorem 31 that there exists a proper circle k^* with proper interior and with center at O such that $\bar{k}$ is within k^*.

THEOREM 34. *Every circle is a proper circle.*

Proof. Suppose there exists a point O such that not every circle with center at O is a proper circle. Then the set of all points is the sum of two mutually exclusive sets S_1 and S_2 such that every point of S_1 lies on a proper circle with center at O and with proper interior, but no point of S_2 lies on any such circle.

It can easily be seen with the aid of Theorem 33 that S_1 contains no limit point of S_2. Let $\bar{P}$ denote some definite point in S_2. By Theorems 9, 10, 11 and 21 the set of all points is a connected domain. It follows by Theorem 23 that there exists a simple continuous arc $O\bar{P}$ from O to $\bar{P}$. There exists a point X which in the order from O to $\bar{P}$ on $O\bar{P}$ is the first point on $O\bar{P}$ that does not belong to S_1. Let OX denote the interval of OP whose end-points are O and X. If $\bar{Y}$ is a point of S_1 then OX has a point O within, and a point X without, the circle with center at O which passes through $\bar{Y}$. Hence this circle contains a point of OX. It follows that for every point $\bar{Y}$ of S_1 there exists a point Y on OX and a motion M such that $M(O)=O$ and $M(Y)=\bar{Y}$. The point-set S_1 is non-compact, otherwise it would* be bounded and its boundary would be a circle with center at O and passing through X, and this circle would be a simple closed curve enclosing O. Hence, there exists in S_1 an infinite set N of distinct points that has no limit point. There exists a set of motions $M_1, M_2, M_3, \ldots.$, all rotations about O, a point P on OX and a set of points $P_1, P_2, P_3, \ldots.$ all lying in S_1 and on OX such that P is a sequential limit point of the sequence $P_1, P_2, P_3, \ldots.$ and such that $M_1(P_1)$, $M_2(P_2)$, $M_3(P_3), \ldots.$ are distinct points of N. Let k denote a definite circle with center at O and lying in S_1 and let C denote a definite point on k.† There exists a sequence of distinct positive integers $n_1, n_2, n_3, \ldots.$ and a point $\bar{C}$ on k such that $\bar{C}$ is the sequential limit point of the sequence $M_{n_1}(C), M_{n_2}(C), M_{n_3}(C), \ldots.$. There exists a motion M such that $M(O)=O$ and $M(C)=\bar{C}$. There exists a region R containing the arc OX. The sequence $n_1, n_2, n_3, \ldots.$ contains an infinite subsequence of distinct integers $j_1, j_2, j_3, \ldots.$ such that $M_{j_1}(P_{j_1})$, $M_{j_2}(P_{j_2}), M_{j_3}(P_{j_3}), \ldots.$ are all without the region $M(R)$. For each i the arc $M_{j_i}(OP_{j_i})$ contains a point $\bar{E}_{j_i}$ on the boundary of $M(R)$. There exists a point $\bar{E}$ on the boundary of $M(R)$ and an infinite sequence of distinct integers $k_1, k_2, k_3, \ldots.$ of the sequence $j_1, j_2, j_3, \ldots.$ such that $\bar{E}$ is the sequential limit point of $\bar{E}_{k_1}, \bar{E}_{k_2}, \bar{E}_{k_3}, \ldots.$. There exists on OX a sequence of points $E_{k_1}, E_{k_2}, E_{k_3}, \ldots.$, all belonging to S_1, such that, for every i, $\bar{E}_{k_i}=M_{k_i}(E_{k_i})$. There exists a point E on OX and an infinite sequence $m_1, m_2, m_3, \ldots.$ of distinct integers belonging to the sequence $k_1, k_2, k_3, \ldots.$ such that E is the sequential limit point of the sequence $E_{m_1}, E_{m_2}, E_{m_3}, \ldots.$. The point $\bar{C}$ is a sequential limit point of the sequence $M_{m_1}(C), M_{m_2}(C), M_{m_3}(C), \ldots.$, the point O is a sequential

*If K is a compact set of points, K' is compact and closed. But every point of K' is in some region. It follows by Theorem 18 and Axiom 5 that K is bounded.

† From here on the present proof bears a certain relationship to an argument of Hilbert's on pp. 407 and 408, *loc. cit.*

limit point of $M_{m_1}(O), M_{m_2}(O), M_{m_3}(O), \ldots$, and the point $\bar{E}$ is a sequential limit point of $M_{m_1}(E_{m_1}), M_{m_2}(E_{m_2}), M_{m_3}(E_{m_3}), \ldots$. It follows that there exists a motion M^* such that $M^*(O)=O$, $M^*(C)=\bar{C}$ and $M^*(E)=\bar{E}$. Since $M^*(O)=M(O)=O$ and $M^*(C)=M(C)$ therefore, by Theorem 32, $M^*(E_{m_1}) =M(E_{m_1}), M^*(E_{m_2})=M(E_{m_2}), \ldots$. But the point E is a sequential limit point of $E_{m_1}, E_{m_2}, E_{m_3}, \ldots$, the point $M(E)$ is a sequential limit point of $M(E_{m_1}), M(E_{m_2}), M(E_{m_3}), \ldots$, and $M^*(E)$ is the sequential limit point of $M^*(E_{m_1}), M^*(E_{m_2}), M^*(E_{m_3}), \ldots$. It follows that $M^*(E)=M(E)$. But $M^*(E)$ is on the boundary of $M(R)$ while $M(E)$ is in $M(R)$. Thus the supposition that there exist points that are not in S_1 has led to a contradiction. It follows that every point distinct from O is on a simple closed curve that encloses O and is a circle with center at O.

THEOREM 35. *There exists an infinite system of circles $k_1, k_2, k_3, \ldots$ such that for every n, k_n is within k_{n+1} and such that every point is within some k_n.*

Proof. Let O denote some definite point. By Theorem 22 there exists a countably infinite set of distinct points $X_1, X_2, X_3, \ldots$ such that $X_1+X_2+X_3+\ldots$ has no limit point. Through the first X_n that is distinct from O there passes a circle k_1 with center at O. Through the first X_n that is without k_1 there is another such circle k_2. This process may be continued. It follows that there exists a sequence $k_1, k_2, k_3, \ldots$ of circles with center at O such that, for every n, k_{n+1} encloses both k_n and the point-set

$$O+X_1+X_2+\ldots+X_n.$$

If Z is a point distinct from O, and k is the circle with center at O that contains Z, there exists a positive integer m such that X_m is without k. The circle k_m encloses Z. The truth of Theorem 35 is therefore established.

THEOREM 36. *The set of all points $\bar{S}$ is a number plane in the sense that between $\bar{S}$ and the set of all sensed pairs of real numbers there is a one-to-one correspondence such that in $\bar{S}$ the point $\bar{P}$ is a sequential limit point of the sequence of points $\bar{P}_1, \bar{P}_2, \bar{P}_3, \ldots$ if and only if $\lim_{n=\infty} x_n=x$ and $\lim_{n=\infty} y_n=y$, where (x, y) is the pair of real numbers corresponding to P and, for every n, (x_n, y_n) is the pair corresponding to $\bar{P}_n$.*

Proof. Let $k_1, k_2, k_3, \ldots$ be a sequence of circles satisfying the requirements of Theorem 35, and let $R_1, R_2, R_3, \ldots$ be their interiors. Let both R_{-1} and R_0 denote the null-set. For every positive n let E_n denote the point-set R_n-R_{n-1}. For every n let $K_{E'_n1}, K_{E'_n2}, K_{E'_n3}, \ldots$ be an infinite sequence

of regions satisfying with respect to the point-set E'_n the requirements of Theorem 19 and such that each of them is a subset of $R_{n+1}-R_{n-2}$. Let $K_1, K_2, K_3, \ldots.$ denote the sequence of regions

$$K_{E'_1 1},\ K_{E'_1 2},\ K_{E'_2 1},\ K_{E'_1 3},\ K_{E'_2 2},\ K_{E'_3 1},\ K_{E'_1 4},\ K_{E'_2 3},\ K_{E'_3 2},\ K_{E'_4 1},\ K_{E'_1 5},\ \ldots.$$

It is easy to see that the sequence $K_1, K_2, K_3, \ldots.$ satisfies, with respect to $\overline{S}$, the requirements of Axiom 1 of F. A. As has already been shown, if in the statement of Axioms 2–8 of F. A. the term *region* is restricted so as to apply only to Jordan regions, the so modified axioms are all fulfilled in $\overline{S}$. It follows* that Theorem 36 is true.

It having been established that $\overline{S}$ is a number plane, one may now follow Hilbert and arrive at the conclusion that if *straight lines*, *congruence*, etc., are defined as he defines them, the so determined geometry is Euclidean or Bolyai–Lobachevskian, according as our group of motions has or has not an invariant subgroup other than the identity. In this sense *every geometry that satisfies the axioms of Σ is either a Euclidean or a Bolyai–Lobachevskian geometry of two dimensions.*

§ 6. *Independence Examples.*

The symbol E_n will be used to denote an example of a system in which Axiom n is false, but all the other axioms of the set Σ are true. In each example E_n, except E_1, use is made of a well-defined space S_n. In every case the *points* of E_n are the ordinary poins of S_n, but the *regions* of the various E_n's are defined in various ways.

E_1. In E_1 the terms *point*, *region* and *motion* have no significance.

E_2. S_2 is Euclidean space of two dimensions. A point-set M is a *region* if and only if $M=I$ or $M=I-P$ where I is the interior of some Jordan curve and P is a point in I. A *motion* is an ordinary two-dimensional motion that does not change sense on any closed curve.

E_3. S_3 is Euclidean space of two dimensions. A *region* is the interior of a Jordan curve whose maximum diameter is equal to or greater than 1. *Motion* has the same significance as in E_2.

E_4. S_4 is the linear continuum $(-\infty \leqq x \leqq \infty)$. A *region* is a segment. A *motion* is a one-dimensional rigid motion that does not reverse order.

E_5. S_5 is a point-set composed of a countably infinite set of equal spheres $K_1, K_2, K_3, \ldots.,$ all lying in a fixed Euclidean space E of three dimensions, such that every K_n is wholly without every other one. A *motion* is a one-to-

* Cf. proof of Theorem 29.

one transformation of S_n into itself which results from first permuting or leaving fixed the spheres of the set and then rotating each sphere about one of its diameters. A set of points M is a *region* if and only if it is one of the two point-sets into which one of the spheres is separated by a closed Jordan curve lying on it.

E_6. S_6 is Euclidean space of three dimensions. A set of points is a *region* if and only if it is the interior of a cube. Motion has its usual significance.

E_7. S_7 is Euclidean space of two dimensions. A set of points is a *region* if and only if it is the interior of a closed Jordan curve. The only motion is the identity transformation.

E_8. $S_8=S_7$. Motion has the same meaning as in E_2. The interior of every simple closed curve is a *region*. Every *region*, with the exception of a certain *region* R_0, is the interior of a simple closed curve. The *region* R_0 is the domain enclosed by the point-set k_1+k_2 described on page 162 of F. A.

E_9. $S_9=S_7$. *Region* the same as in E_7. Every one-to-one continuous transformation of S_9 into itself that does not change sense is a *motion*.

E_{10}. Same space and same *regions* as in E_7. Select a system of rectangular coordinates. Let $\bar{M}$ denote the transformation of S_{10} into itself represented by the equations $x'=2x$, $y'=2y$. A transformation is a motion if and only if it can be expressed in the form $M_1M_2M_3\ldots.M_n$ where $M_1, M_2, \ldots, M_n$ is a finite set of transformations, distinct or otherwise, such that for each $i(1\leqq i\leqq n) M_2$ is either the transformation $\bar{M}$ or some rigid motion that leaves sense invariant.

E_{11}. Same space and same *regions* as in E_{10} Let $\bar{M}$ have the same meaning as in E_{10} and let $\bar{M}^{-1}$ represent the inverse of $\bar{M}$. A one-to-one transformation of S_{11} into itself is a motion if and only if it is identical with $\bar{M}$ or with $\bar{M}^{-1}$, or with some ordinary two-dimensional rigid motion that leaves sense invariant.

E_{12}. Same space and same *regions* as in E_7. A *motion* is a one-to-one continuous transformation of S_{12} into itself that carries lines into lines and preserves distances.

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